Combining Dependent Sources of Information Within the Framework of Evidence Theory

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Abstract. The paper introduces and discusses a family of conjunctive information aggregation operators within the framework of evidence theory based on the use of copulas and t-norms. Such operators allow modeling the dependence of information sources and aggregating many sources. It is shown that the introduced operators will satisfy to a certain extent the general requirements (desirable properties) imposed on the rules of combination in the theory of evidence. A new type of conjunctive aggregation for jointly consonant bodies of evidence is explored in more detail. The choice of one or another t-norm or copula in such a conjunctive rule can be based on solving a multi-criteria choice problem with respect to several aggregation quality functionals.

Keywords: Evidence theory · Conjunctive aggregation rules · Dependent information sources · t-norms · Copulas

1 Introduction

In evidence theory [3,22] information is represented in the form of so-called bodies of evidence. Each body of evidence consists of a set of subsets of membership of the true alternative (the so-called focal elements) and degrees of belief (mass functions) in such events. The masses of focal elements can be interpreted as probabilities of random sets. In classical evidence theory [22], all operations on bodies of evidence (e. g., aggregation or assessment of inconsistency) are defined formally without taking into account the significance of the focal elements themselves, the degree of their inclusion or intersection, the dependence of their masses, etc.

The most popular aggregation rules [21] (e. g., Dempster's and Yager's rules) have a conjunctive form, in which the mass of a new body of evidence on a set A is calculated as the sum of the products of the masses of focal elements whose intersection is equal A. The "vanishing mass effect" is observed for such rules. If the number of aggregated sources of information (bodies of evidence) is large, then the masses of the focal elements of the new body can be small even in the case of intersection of focal elements with large masses, which is counterintuitive.

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Moreover, the form of the popular Dempster and Yager rules as sums of products of the masses of the aggregated bodies of evidence assumes the independence of the sources of information. If the sources of information are not independent, then this dependence can be modeled using the "tuning" of the mass function aggregation operator [12]. Copulas [19], t-norms and conorms [15] are popular aggregation operators.

Thus, in a recent work [20], simple bodies of evidence about a pattern's membership in a particular class were conjunctively aggregated to construct a new evidence-based classifier, that takes into account various dependencies between such sources. This dependence was modeled using Frank's parametric copula [11], whose parameter was the correlation coefficient (such a correlation model was discussed in [10]). This method of modeling dependence can be called a priori.

Aggregation of dependent Gaussian Random Fuzzy Numbers in the evidence theory framework was considered in a recent paper [5].

In this paper, we introduce and discuss a family of operators for aggregating bodies of evidence of a conjunctive type (i. e., new focal elements are constructed at the intersection of focal elements of the aggregated bodies of evidence), but based on the application of copulas and t-norms to the masses of the aggregated bodies of evidence (in the case of t-norms, such rules will be called T-conjunctive). It is shown that such operators will satisfy to a certain extent the general requirements (desirable properties) imposed on aggregation operators. The axiomatic approach to constructing aggregation rules is widely represented in evidence theory [13,16]. In this paper, the noted desirable properties are analyzed for T-conjunctive aggregation rules. In addition, T-conjunctive aggregation of jointly consonant bodies of evidence is explored in more detail. The paper uses the a posteriori approach as a way of modeling the dependence between information sources. The qualitative characteristics of the bodies of evidence obtained as a result of aggregation are analyzed in this case. It is shown in a numerical example that the specification of a particular t-norm or copula in a T-conjunctive rule can be implemented as a solution to a multi-criteria choice problem with respect to several aggregation quality functionals.

The rest of the article has the following structure. Section 2 provides the necessary background on evidence theory, t-norms and copulas. Section 3 discusses desirable properties of conjunctive aggregation rules and modeling of dependencies between bodies of evidence. Section 4 discusses in detail the T-conjunctive aggregation of jointly consonant bodies of evidence. Section 5 provides a numerical example and Sect. 6 draws some conclusions.

2 Necessary Background of Evidence Theory

2.1 The Concept of a Body of Evidence

Let us recall some provisions of the theory of evidence. We will adhere to the interpretation of this theory using random sets [4]. Let X be some set, 2^X be the set of all subsets from X, $\mathcal{A}$ be some finite set of subsets from X (the set of focal elements). The set $\mathcal{A}$ can be considered as a set of values of random sets ω. Then the function of sets $m(A) = \begin{cases} P\{\omega = A\}, & A \in \mathcal{A}, \\ 0, & A \notin \mathcal{A}, \end{cases}$ defined on 2^X is called the basic probability assignment (bpa) or the mass function, $\sum_{A\in\mathcal{A}} m(A) = 1$. A pair $F = (\mathcal{A}, m)$ is called a body of evidence on the set X. If $m(\emptyset) = 0$, then the body of evidence is called normal. Let $\mathcal{F}(X)$ be the set of all bodies of evidence on X. The belief functions Bel and the plausibility functions Pl can be mutually one-to-one assigned to the body of evidence $F = (\mathcal{A}, m)$ using the formulas

$$Bel(A) = P\{\omega \subseteq A\} = \sum_{B\subseteq A} m(B), \ \ Pl(A) = P\{\omega \cap A \neq \emptyset\} = \sum_{B\cap A\neq\emptyset} m(B),$$

$A \in 2^X$. In particular, the function $Pl(x) = \sum_{B:x\in B} m(B)$ is called contour.

The following belief structures are distinguished among the bodies of evidence:

- a categorical body of evidence of the type $F_A = (\{A\}, 1)$, $A \in \mathcal{A}$, consisting of only one focal element with unit mass; in particular, a categorical body of evidence $F_X = (\{X\}, 1)$ is called vacuous;
- consonant body of evidence, if $A \subseteq B$ or $B \subseteq A$ for any $A, B \in \mathcal{A}$.

The contour function of a consonant body of evidence (and only for them) coincides with the possibility distribution function, which can be viewed as a membership function of a fuzzy set.

Below in the numerical example we will consider the bodies of evidence on $X \subseteq \mathbb{R}$. In this case we will we will use the evidential cardinality [8]

$$H(F) = \sum_{A\in\mathcal{A}} m(A)\lambda(A)$$

or the normalized evidential cardinality $H_0(F) = H(F)/\lambda(X) \in [0, 1]$ (if X is a set with finite measure) to estimate the degree of uncertainty of the information given by the body of evidence $F = (\mathcal{A}, m)$ on $X \subseteq \mathbb{R}$, where $\lambda(A)$ is the measure of the set A on $\mathbb{R}$. Note that the evidential cardinality coincides with the cardinality of the corresponding fuzzy set for consonant bodies of evidence:

$$\sum_{A\in\mathcal{A}} m(A)\lambda(A) = \int_{\mathbb{R}} Pl(x)\, dx.$$

In addition, if $\mu_{\tilde{A}}$ is a membership function of a fuzzy set $\tilde{A}$, then we will estimate its fuzziness using the formula

$$Fuz(\tilde{A}) = 1 - \frac{1}{\lambda(X)} \int_{\mathbb{R}} |2\mu_{\tilde{A}}(x) - 1|\, dx \in [0, 1].$$

The convex sum of the bodies of evidence $F_1 = (\mathcal{A}_1, m_1)$, $F_2 = (\mathcal{A}_2, m_2) \in \mathcal{F}(X)$ with the coefficients $\alpha, \beta \in [0, 1]$, $\alpha + \beta = 1$ is called the body of evidence

$F = (\mathcal{A}, m)$, where , $m(A) = \alpha m_1(A) + \beta m_2(A)$, $A \in \mathcal{A}$. Then any body of evidence $F = (\mathcal{A}, m)$ can be represented in the form

$$F = \sum_{A \in \mathcal{A}} m(A) F_A.$$

In particular, the body of evidence of the type

$$F_A^\alpha = \alpha F_A + (1 - \alpha) F_X, \ \alpha \in [0, 1]$$

is called simple.

Different types of inclusion relations are considered on the set $\mathcal{F}(X)$ [7]. Let $F_i = (\mathcal{A}_i, m_i) \in \mathcal{F}(X)$, Bel_i, Pl_i be the belief and plausibility functions corresponding to F_i, $i = 1, 2$. We will distinguish the following inclusion relations:

a) the F_1 is weakly included in the F_2 (designation: $F_1 \sqsubseteq_w F_2$) if $Pl_1(A) \le Pl_2(A)$ ($\Leftrightarrow Bel_1(A) \ge Bel_2(A)$) $\forall A \in 2^X$;
b) the F_1 is strongly included in the F_2 (designation: $F_1 \sqsubseteq_s F_2$) if there is a non-negative matrix $W = (w(A, B))_{A \in \mathcal{A}_1, B \in \mathcal{A}_2}$:

$$\sum_{A \in \mathcal{A}_1} w(A, B) = m_2(B) \ \forall B \in \mathcal{A}_2, \ \sum_{B \in \mathcal{A}_2} w(A, B) = m_1(A) \ \forall A \in \mathcal{A}_1$$

and $w(A, B) > 0 \Rightarrow A \subseteq B$;
c) the F_1 is included in the F_2 with equal power (designation: $F_1 \sqsubseteq_e F_2$) if $\forall A \in \mathcal{A}_1 \ \exists ! B \in \mathcal{A}_2$: $A \subseteq B$ and $\forall B \in \mathcal{A}_2 \ \exists ! A \in \mathcal{A}_1$: $A \subseteq B$, $m_1(A) = m_2(B)$ in these cases.

It is clear that $F \sqsubseteq_s F_X \ \forall F \in \mathcal{F}(X)$. If $F_1 \sqsubseteq_s F_2$, then the body of evidence F_1 is called a specialization of the body of evidence F_2, and F_2 is called a generalization of F_2. It is know [7] that $F_1 \sqsubseteq_e F_2 \Rightarrow F_1 \sqsubseteq_s F_2 \Rightarrow F_1 \sqsubseteq_w F_2$, but the converse is not true. But for consonant bodies of evidence, strong and weak inclusions coincide.

2.2 Rules for Aggregating Bodies of Evidence

The toolkit for combining (aggregating) bodies of evidence is widely represented in evidence theory [21]. The generalized conjunctive rule of combining $\otimes : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathcal{F}(X)$ is the most popular of them. It is introduced according to the following scheme [1]: $F = (\mathcal{A}, m) = F_1 \otimes F_2$ and $F_1 = (\mathcal{A}_1, m_1)$, $F_2 = (\mathcal{A}_2, m_2) \in \mathcal{F}(X)$, where

$$m(A) = \sum_{B \cap C = A} \tilde{m}(B, C),$$

and the sets function $\tilde{m} : 2^X \times 2^X \to [0, 1]$ satisfies the conditions:

$$\sum_{C \in 2^X} \tilde{m}(B, C) = m_1(B), \ \sum_{B \in 2^X} \tilde{m}(B, C) = m_2(C), \ B, C \in 2^X.$$

If the sources of information are independent, then $\tilde{m}(B,C) = m_1(B)m_2(C)$, $\forall B, C \in 2^X$ and we get the non-normalized Dempster rule $\otimes_{ND}$:

$$m_{ND}(A) = \sum_{B \cap C = A} m_1(B)m_2(C),\ \forall A \in 2^X.$$

The value

$$m_{ND}(\emptyset) = \sum_{B \cap C = \emptyset} m_1(B)m_2(C) \in [0,1]$$

characterizes the degree of conflict between the sources of information described by the bodies of evidence F_1 and F_2. If the sources of information are not absolutely conflicting (i. e. $m_{ND}(\emptyset) < 1$), then the bpa of the aggregated body of evidence can be normalized

$$m_D(A) = \frac{m_{ND}(A)}{1 - m_{ND}(\emptyset)}\ \ \forall A \in 2^X \backslash \emptyset,\ \ m_D(\emptyset) = 0.$$

As a result, we obtain the classical Dempster rule $\otimes_D$ [3].

3 Conjunctive Rules for Combining Dependent Bodies of Evidence

3.1 Desirable Properties of Conjunctive Combination Rules

The following properties are usually required of conjunctive combination rules $\otimes : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathcal{F}(X)$ [2,9]:

A1. $F \otimes F_X = F\ \forall F \in \mathcal{F}(X)$ (i. e. F_X is a neutral element);
A2. $F_1 \otimes F_2 = F_2 \otimes F_1\ \forall F_1, F_2 \in \mathcal{F}(X)$ (commutativity);
A3. $F_1 \otimes (F_2 \otimes F_3) = (F_1 \otimes F_2) \otimes F_3\ \forall F_1, F_2, F_3 \in \mathcal{F}(X)$ (associativity);
A4. $F \otimes F = F$ (idempotency);
A5. If $F_1, F_2 \in \mathcal{F}(X)$ are strongly consistent (i.e. $A_1 \cap A_2 \neq \emptyset\ \forall A_i \in \mathcal{A}_i$, $i = 1,2$) and $F_i \sqsubseteq F_i'$, $i = 1,2$, then $F_1 \otimes F_2 \sqsubseteq F_1' \otimes F_2'$ (information monotonicity), where $\sqsubseteq$ is some type of inclusion;
A6. If $F_1 = F_A^{\alpha_1}$,$F_2 = F_A^{\alpha_2}$,$F = F_B^{\beta} \in \mathcal{F}(X)$, $0 \le \alpha_1 \le \alpha_2 \le 1$, then $m_{F_1 \otimes F}(A \cap B) \le m_{F_2 \otimes F}(A \cap B)\ \forall \beta \in [0,1]$.

Note that property A1 always holds for the generalized conjunctive combination rule, since in this case we have $\tilde{m}(B,C) = \begin{cases} 0, & C \neq X, \\ m(B), & C = X \end{cases}\ \forall B$ $\Rightarrow m_{F \otimes F_X}(A) = \sum_{B \cap C = A} \tilde{m}(B,C) = \tilde{m}(A,X) = m(A)\ \forall A$.

The optimism property for conjunctive rules follows from properties A1 and A5: if $F_1, F_2 \in \mathcal{F}(X)$ are strongly jointly consistent, then $F_1 \otimes F_2 \sqsubseteq_q F_i$, $i = 1,2$ (since, for example, $F_1 \otimes F_2 \sqsubseteq_q F_1 \otimes F_X = F_1$), where $q \in \{s, w\}$. Note also that strongly jointly consistent bodies of evidence are absolutely non-conflicting (i. e. $m_{ND}(\emptyset) = 0$) and this property is preserved when moving to their generalizations.

Remark 1. Properties A1–A4 are algebraic and define a commutative idempotent monoid.

All the above properties, except idempotency, are satisfied with the Dempster's rule (but the optimism property is also satisfied). In general, the property of idempotency is the most critical of the indicated properties. It does not hold for many popular combination rules. A discussion of the idempotency condition and the construction of idempotent rules can be found in [6,14]. In addition to those indicated, other desirable (usually non-algebraic) ones are introduced [2,9].

3.2 Modeling Dependency in a Conjunctive Rule

We consider conjunctive rules for combining bodies of evidence $F_1 = (\mathcal{A}_1, m_1)$ and $F_2 = (\mathcal{A}_2, m_2)$ defined on X of the form $F = F_1 \otimes_G F_2 = (\mathcal{A}, m^{(G)})$, where$\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2 = \{A_1 \cap A_2 \neq \emptyset : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}$ and

$$m^{(G)}(A) = C_G \sum_{B \cap C = A} G\left(m_1(B), m_2(C)\right), \;\; A \in \mathcal{A}, \tag{1}$$

$G : [0,1]^2 \to [0,1]$ is some aggregation operator [12], $C_G^{-1} = C_G^{-1}(F_1, F_2) = \sum_{B \cap C \neq \emptyset} G\left(m_1(B), m_2(C)\right)$ is a normalization coefficient. The operator $G\left(m_1(B), m_2(C)\right)$ defines some dependence between random events $\omega_1 = B$ and $\omega_2 = C$, $m_1(B) = P\{\omega_1 = B\}$, $m_2(C) = P\{\omega_2 = C\}$. In particular, if random events $\omega_1 = B$ and $\omega_2 = C$ are independent, then $G\left(m_1(B), m_2(C)\right) = m_1(B) m_2(C)$ and we obtain the classical conjunctive rule. Rules of the form (1) will be called G-conjunctive.

Operator (1) will not be defined if: 1) the sources of information are absolutely conflicting, i. e. $B \cap C = \emptyset \; \forall B \in \mathcal{A}_1, \; C \in \mathcal{A}_2$; 2) $C_G^{-1} = 0$, i. e. $G\left(m_1(B), m_2(C)\right) = 0 \; \forall B \in \mathcal{A}_1, \; C \in \mathcal{A}_2$ (for example, for $F_1 = F_{A_1}^{0.5}$, $F_2 = F_{A_2}^{0.5}$ $\in \mathcal{F}(X)$ and $G = \max\{x + y - 1, 0\}$).

The most general model of dependence between random events is the copula $C : [0,1]^2 \to [0,1]$, which satisfies the conditions:

C1. $C(x,y) + C(u,v) \geq C(x,v) + C(u,y) \; \forall x,y,u,v \in [0,1], \; x \leq u, \; y \leq v$;
C2. $C(x,0) = C(0,x) = 0 \; \forall x \in [0,1]$;
C3. $C(x,1) = C(1,x) = x \; \forall x \in [0,1]$.

It is known that any copula is limited by functions (Fréchet – Hoeffding limits):

$$\max\{x + y - 1, 0\} \leq C(x,y) \leq \min\{x,y\}.$$

The concept of a copula is closely related to the concept of a t-norm $T : [0,1]^2 \to [0,1]$ that satisfies the conditions:

T1. $T(x,y) = T(y,x) \; \forall x,y \in [0,1]$;
T2. $T(x, T(y,z)) = T(T(x,y), z) \; \forall x,y,z \in [0,1]$;
T3. $T(x,y) \leq T(x,z) \; \forall x,y,z \in [0,1]$ if $y \leq z$;
T4. $T(x,1) = x \; \forall x \in [0,1]$.

It is known [15] that any associative copula is a continuous t-norm, and a t-norm is a copula if and only if it satisfies the Lipschitz condition with constant equal to 1.

Note that a T-conjunctive operator $\otimes_T$ with an arbitrary t-norm T will certainly satisfy conditions A1, A2. Condition A3 (associativity) will not be satisfied in the general case, but the property of quasi-associativity [23] will be satisfied, i. e. the operator can be represented as a sequential execution of associative operations. Property A4 (idempotency) will only hold in special cases, such as when $T(x,y) = \min(x,y)$ and $A \cap B = \emptyset \; \forall A, B \in \mathcal{A}$ (strongly inconsistent bodies of evidence). Property A5 will obviously be fulfilled for the relation $\sqsubseteq_e$. Property A6 will hold, for example, for the product $T_P(x,y) = xy$ and minimum $T_M(x,y) = \min\{x,y\}$ t-norms. But it will not hold for the Łukasiewicz t-norm $T_L(x,y) = \max\{x+y-1, 0\}$.

The concept of t-norm is generalized by induction to the l-ary operation

$$T^{(l)}(x_1,\ldots,x_l) = T(T^{(l-1)}(x_1,\ldots,x_{l-1}), x_l),$$

$T^{(2)}(x_1,x_2) = T(x_1,x_2)$, $l = 3,4,\ldots$ due to associativity. For example,

$$T_L^{(l)}(x_1,\ldots,x_l) = \max\{x_1 + \ldots + x_l - (l-1), 0\}.$$

Example 1. Suppose two analysts give a forecast of the stock price at the end of the period. The first states that the stock price will be in the range of $[40,55]$ with a confidence level of 0.3 or in the range of $[40,55]$ with a confidence level of 0.7. The second gives a forecast: the price per share will be in the range of $[45,50]$ with a confidence level of 0.4 or in the range of $[35,55]$ with a confidence level of 0.6. These predictions can be written using two bodies of evidence $F_1 = (\mathcal{A}_1, m_1)$ and $F_2 = (\mathcal{A}_2, m_2)$

$$F_1 = 0.3F_{[40,50]} + 0.7F_{[40,55]}, \quad F_2 = 0.4F_{[45,50]} + 0.6F_{[35,55]},$$

defined on the $X = [35,55]$. These bodies of evidence are consonant and absolutely non-conflicting, i. e. $B \cap C \neq \emptyset \; \forall B \in \mathcal{A}_1, \forall C \in \mathcal{A}_2$.

We apply the T-conjunctive operator for $T = T_P$ (in this case, we get Dempster's rule), $T = T_M$ and $T = T_L$. As a result, we obtain new bodies of evidence $F^{(P)}$, $F^{(M)}$, and $F^{(L)}$ respectively:

$$F^{(P)} = 0.4F_{[45,50]} + 0.18F_{[40,50]} + 0.42F_{[40,55]},$$

$$F^{(M)} = \frac{7}{16}F_{[45,50]} + \frac{3}{16}F_{[40,50]} + \frac{6}{16}F_{[40,55]}, \quad F^{(L)} = 0.25F_{[45,50]} + 0.75F_{[40,55]}.$$

We see that these aggregations even have different numbers of focal elements. But they will all be consonant. Therefore, for them, the contour function will be the membership function of the fuzzy set of the prognostic value of shares. Let us find the values of the uncertainty measure H_0 and the degree of fuzziness Fuz. The results are given in Table 1. We see that the least fuzziness is obtained when aggregating using the Łukasiewicz t-norm, and the least uncertainty is obtained when aggregating using the t-norm $T_M = \min$.

Table 1. The values of the uncertainty H_0 and the fuzziness Fuz.

body of evidence	H_0	Fuz
$F^{(P)}$	0.505	0.66
$F^{(M)}$	0.484	0.656
$F^{(L)}$	0.625	0.5

4 *T*-Conjunctive Aggregation of Jointly Consonant Simple Bodies of Evidence

In a number of applications, it is necessary to aggregate several jointly consonant simple bodies of evidence. For example, such a need arises in the problem of evidential regression (dependence recovery) [17,18].

Below in Sect. 5 we will consider a corresponding numerical example. Therefore, let us examine this situation in more detail.

We call simple bodies of evidence $F_{A_k}^{\alpha_k}$, $k = 1, \ldots, l$ jointly consonant if there exists a permutation of indices such that $A_{i_1} \subseteq \ldots \subseteq A_{i_l} \subseteq X$. Note that for a consonant body of evidence with focal elements $\emptyset = A_{i_0} \subseteq A_{i_1} \subseteq \ldots \subseteq A_{i_l} \subseteq A_{i_{l+1}} = X$, the contour function will be equal to

$$Pl(x) = \sum_{k=s}^{l+1} m(A_{i_k}) \; \forall x \in A_{i_s} \backslash A_{i_{s-1}}, \;\; s = 1, \ldots, l+1. \tag{2}$$

The following proposition and its corollaries show how the mass function and the contour function will be found in the general case of T-conjunctive aggregation of l sources and in the special cases of the most popular t-norms.

Proposition 1. *The body of evidence* $F^{(T)} = (\mathcal{A}, m_T) = \underset{k=1}{\overset{l}{\otimes_T}} F_{A_k}^{\alpha_k}$ *obtained as a result of the T-conjunctive aggregation of jointly consonant simple bodies of evidence* $F_{A_k}^{\alpha_k}$, $k = 1, \ldots, l$ *will be consonant,* $\mathcal{A} = \{A_1, \ldots, A_l, X\}$. *Moreover, if* $\emptyset = A_{i_0} \subseteq A_{i_1} \subseteq \ldots \subseteq A_{i_l} \subseteq A_{i_{l+1}} = X$, *then (assuming* $\alpha_{i_0} = 0$*) we have for* $k = 1, \ldots, l-1$

$$\begin{gathered} m_T(A_{i_k}) = C_T \sum_{\substack{(\beta_{i_{k+1}}, \ldots, \beta_{i_l}): \\ \beta_{i_s} = \alpha_{i_s} \vee 1 - \alpha_{i_s}}} T\left(1 - \alpha_{i_0}, \ldots, 1 - \alpha_{i_{k-1}}, \alpha_{i_k}, \beta_{i_{k+1}}, \ldots, \beta_{i_l}\right), \\ m_T(A_{i_l}) = C_T T\left(1 - \alpha_{i_1}, \ldots, 1 - \alpha_{i_{l-1}}, \alpha_{i_l}\right), \\ m_T(X) = C_T T\left(1 - \alpha_{i_1}, \ldots, 1 - \alpha_{i_l}\right), \end{gathered} \tag{3}$$

where

$$C_T^{-1} = T\left(1 - \alpha_{i_1}, \ldots, 1 - \alpha_{i_l}\right) + T\left(1 - \alpha_{i_1}, \ldots, 1 - \alpha_{i_{l-1}}, \alpha_{i_l}\right) +$$

$$+\sum_{k=1}^{l-1}\sum_{\substack{(\beta_{i_{k+1}},\ldots,\beta_{i_l}):\\ \beta_{i_s}=\alpha_{i_s}\vee 1-\alpha_{i_s}}} T\left(1-\alpha_{i_0},\ldots,1-\alpha_{i_{k-1}},\alpha_{i_k},\beta_{i_{k+1}},\ldots,\beta_{i_l}\right).$$

Corollary 1. *We have for* $T_P(x_1,\ldots,x_l)=x_1\ldots x_l$

$$m_P\left(A_{i_k}\right)=\left(1-\alpha_{i_0},\right)\ldots\left(1-\alpha_{i_{k-1}}\right)\alpha_{i_k},\ k=1,\ldots,l+1,\ \left(\alpha_{i_{l+1}}=1\right).$$

In addition, we have $Pl_P(x)=\left(1-\alpha_{i_0},\right)\ldots\left(1-\alpha_{i_{k-1}}\right)$ *for* $x\in A_{i_k}\backslash A_{i_{k-1}}$, $k=1,\ldots,l+1$.

Corollary 2. *If* $0=\alpha_{i_0}\leq\alpha_{i_1}\leq\ldots\leq\alpha_{i_l}\leq\alpha_{i_{l+1}}=1$, $\alpha_{i_{k-1}}+\alpha_{i_k}\geq 1$, $k=2,\ldots,l$ *is true, then we have for* $T_M(x_1,\ldots,x_l)=\min\{x_1,\ldots,x_l\}$

$$m_M\left(A_{i_1}\right)=C_M\left(\alpha_{i_1}+\sum_{s=0}^{l-2}2^s\left(1-\alpha_{i_{s+2}}\right)\right),\ m_M\left(X\right)=C_M\left(1-\alpha_{i_l}\right),$$

$$m_M\left(A_{i_k}\right)=C_M\left(\left(1-\alpha_{i_{k-1}}\right)+\sum_{s=0}^{l-k}2^s\left(1-\alpha_{i_{k+s+1}}\right)\right),\ k=2,\ldots,l,$$

where $C_M^{-1}=1+\sum\limits_{k=2}^{l}\left(1-\alpha_{i_k}\right)+\sum\limits_{k=1}^{l}\sum\limits_{s=0}^{l-k}2^s\left(1-\alpha_{i_{k+s+1}}\right)=2^l-1-\sum\limits_{k=2}^{l}2^{k-1}\alpha_{i_k}$.

In addition, we have

$$Pl_M(x)=\begin{cases}C_M\left(1-\alpha_{i_l}\right), & x\in X\backslash A_{i_l},\\ C_M\left(2^{l-k+1}-\alpha_{i_{k-1}}-\sum\limits_{s=0}^{l-k}2^s\alpha_{i_{k+s}}\right), & x\in A_{i_k}\backslash A_{i_{k-1}},\ k=2,\ldots,l,\\ 1, & x\in A_{i_1}.\end{cases}$$

Corollary 3. *If* $\alpha_i\geq 1-\frac{p}{l}$, $i=1,\ldots,l$, $p\in\{1,\ldots,l-1\}$ *is true, then we have for* $T_L(x_1,\ldots,x_l)=\max\{x_1+\ldots+x_l-l+1,0\}$*:* $m_L\left(X\right)=m_L\left(A_{i_k}\right)=0$ $\forall k=p+2,\ldots,l$. *In particular, if* $p=1$, *then* $m_L\left(X\right)=m_L\left(A_{i_k}\right)=0$ $\forall k=3,\ldots,l$, $m_L\left(A_{i_1}\right)=C_L\left(|\alpha|+\sum\limits_{s=2}^{l}\max\left\{\left|\alpha^{(s)}\right|-l+1,0\right\}\right)$, $m_L\left(A_{i_2}\right)=C_L\max\left\{\left|\alpha^{(1)}\right|-l+1,0\right\}$, *where* $|\alpha|=\sum\limits_{k=1}^{l}\alpha_{i_k}$, $\left|\alpha^{(s)}\right|=1-\alpha_{i_s}+\sum\limits_{k=1,k\neq s}^{l}\alpha_{i_k}$, $s=1,\ldots,l$, $C_L^{-1}=|\alpha|+\sum\limits_{s=1}^{l}\max\left\{\left|\alpha^{(s)}\right|-l+1,0\right\}$. *In this case we have*

$$Pl_L(x)=\begin{cases}0, & x\notin A_{i_2},\\ C_L\max\left\{\left|\alpha^{(1)}\right|-l+1,0\right\}, & x\in A_{i_2}\backslash A_{i_1},\\ 1, & x\in A_{i_1}.\end{cases}$$

5 Numerical Example

Let us give an example of a T-conjunctive aggregation of $l = 5$ jointly consonant simple bodies of evidence $F_{A_k}^{\alpha_k}$, $k = 1, \ldots, 5$, given on a number line. A similar problem is relevant, for example, when finding bodies of evidence for dependence coefficients using evidential regression methods [17,18].

Let $X = [-3, 3]$, $A_k = [-0.5k, 0.5k]$, $\alpha_k = 0.1 + 0.2k$, $k = 1, \ldots, 5$, $F^{(T)} = \bigotimes_{k=1}^{5}{}_T F_{A_k}^{\alpha_k}$. The graphs of contour functions Pl_T for the t-norms $T = T_P$, $T = T_M$ and the Schweizer – Sklar t-norm

$$T_{L,s}(x_1, \ldots, x_l) = (\max\{x_1^s + \ldots + x_l^s - l + 1, 0\})^{\frac{1}{s}}, \;\; s \neq 0$$

for $s = -0.5$ calculated using formulas (2) and (3), are shown in Fig. 1.

It is known [15] that, $\lim\limits_{s \to -\infty} T_{L,s} = T_M$, $\lim\limits_{s \to 0} T_{L,s} = T_P$. Note that for the t-norm $T_{L,s}$ for $s > 0.7$ aggregation is not defined, since $C_{T_{L,s}} = 0$.

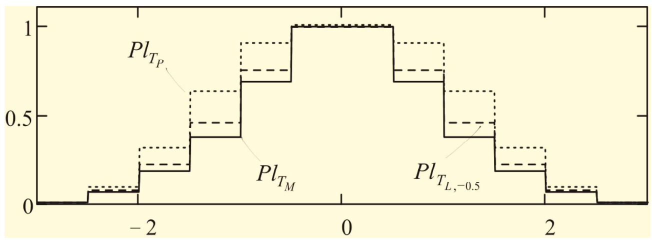

Fig. 1. Contour functions of T-conjunctive aggregation for $T = T_P$, $T = T_M$, $T_{L,-0.5}$.

It is easy to see that we will obtain bodies of evidence with varying degrees of uncertainty depending on the choice of the aggregation function T: $H(F^{(P)}) = 2.95$, $H(F^{(M)}) = 2.31$, $H(F^{(L,-0.5)}) = 2.5$.

The selection of a specific t-norm from a certain class can be carried out as a solution to a multi-criteria problem of analyzing the qualitative characteristics of the aggregation operator. For example, these could be the characteristics of uncertainty H_0 and fuzziness Fuz in our example. The points $\left(Fuz(F^{(T)}), H_0(F^{(T)})\right)$ for t-norms $T = T_P$, T_M, $T_{L,s}$ and $s = -1$, -0.5, 0.1, 0.3, 0.5 are shown in Fig. 2. It can be seen that T_M-aggregation is the best in this example with respect to the uncertainty functional. $T_{L,0.5}$-aggregation will be the best with respect to the fuzziness functional.

The best solution to a two-criterion problem on the Pareto frontier will depend on the chosen solution method. For example, the best choice for a linear convolution of criteria $Con_\theta(F^{(T)}) = Fuz(F^{(T)}) + \theta H_0(F^{(T)})$, $\theta > 0$ would be the solution

$$\arg\min_T Con_\theta(F^{(T)}) = \begin{cases} T_M, & \theta \leq 1.306, \\ T_{L,0.5}, & \theta > 1.306. \end{cases}$$

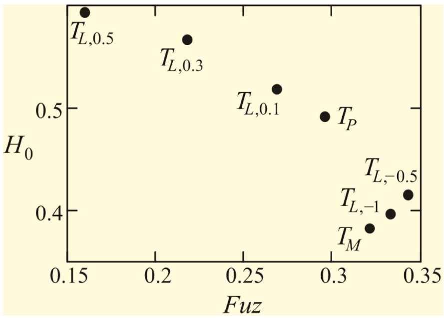

Fig. 2. Solution points $\left(Fuz(F^{(T)}), H_0(F^{(T)})\right)$ for different t-norms.

6 Conclusion

A rule for conjunctive aggregation of bodies of evidence, based on the application of aggregation functions to masses of bodies of evidence, is introduced in this paper. This design can model the dependency between information sources. In addition, it can be useful for aggregating many bodies of evidence. It is shown that the proposed operator will satisfy several desirable properties. A new method for aggregating jointly consonant bodies of evidence is discussed in detail in the case where t-norms are used. The example shows that the choice of the mass aggregating operator in such a conjunctive rule can be based on solving the problem of multi-criteria choice with respect to several aggregation quality functionals.

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