Threshold Aggregation of Fuzzy Data Using Fuzzy Cardinalities of a Set of Fuzzy Estimates *

Alexander Lepskiy[1][0000−0002−1051−2857]

Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow, 101000, Russia,
alepskiy@hse.ru
https://www.hse.ru/en/org/persons/10586209

Abstract. The problem of threshold ranking of alternatives represented by vector fuzzy estimates of a three-gradation scale (high, medium, low) is considered in the article. A generalization of the corresponding procedure for fuzzy estimates is proposed, based on the calculation of the fuzzy cardinalities of a set of fuzzy estimates for each gradation. We discuss the general properties that the fuzzy cardinality of a set of fuzzy estimates must satisfy. The proposed method is illustrated by the example of ranking articles based on their assessments by reviewers and considering information about the degree of confidence of reviewers in their assessments. This information from reviewers is converted into vectors of three-grade fuzzy ratings for each article. These alternatives are then ranked according to the developed methodology. The differences between the results of such ranking and the ranking of unblurred estimates and/or after applying alternative methods are discussed.

Keywords: Threshold rule · Fuzzy cardinality of fuzzy estimates · Ranking of vector fuzzy alternatives.

1 Introduction

Some applied problems of ranking vector scores of alternatives assume that low scores for one criterion cannot be compensated by high scores for other criteria. The corresponding rules for aggregating preferences for the purpose of ranking alternatives are called non-compensatory. The threshold rule for ranking vector alternatives in m-gradation scales is one of the popular rules of this type [1,2]. It is known [2] that the application of this rule comes down to the lexicographic ordering of the cardinality vectors of estimates of different gradations.

If the decision maker does not have absolute confidence in the belonging of a particular assessment of an alternative to a certain gradation, then such a situation can be modeled using some uncertainty tools. This can be modeled

* This article is an output of a research project "Study of the models and methods of decision-making under conditions of deep uncertainty. Extension and testing of developed methods" implemented as part of the Basic Research Program at the National Research University Higher School of Economics (HSE University).

using fuzzy set theory. Thus, the problem of developing a method for threshold ranking of vector fuzzy estimates of alternatives may be relevant.

A method for solving this problem based on calculating a certain measure of proximity between all fuzzy estimates of a certain gradation and the reference number (fuzzy or unblurred) for a given gradation was proposed in a recent work [5]. This approach can be called metric.

In this paper, another approach will be proposed, which is based on calculating the fuzzy cardinality of a set of fuzzy estimates of each gradation.

This approach will be illustrated by the example of ranking articles/reports based on the results of their evaluation by reviewers and considering the degree of confidence of the reviewers in their decision.

The rest of the article is structured as follows. The non-fuzzy threshold aggregation problem is formulated in Section 2. The procedure for calculating the fuzzy cardinality of three-grade fuzzy estimates is described in Section 3. Procedures for comparing cardinality vectors of fuzzy estimates of different gradations and ranking alternatives are discussed in Section 4. A numerical example of ranking alternatives based on the developed methodology is discussed in Section 5. Some preliminary conclusions are given in Section 6.

2 Threshold Aggregation Problem. Unblurred Case

Let us assume that the alternatives are represented by n-dimensional vectors $\mathbf{x} = (x_1, \ldots, x_n)$ on a three-gradation scale, i. e. $x_i \in \{1, 2, 3\}$. Let X be a set of alternatives of this type that need to be ranked. The ranking procedure is carried out using an aggregation operator $\varphi_n = \varphi : X \to \mathbb{R}$, which must satisfy the following axioms [2]:

1. Pareto-domination: if $\mathbf{x}, \mathbf{y} \in X$ and $x_i \geq y_i \quad \forall i, \exists s : \ x_s > y_s$, then $\varphi(\mathbf{x}) > \varphi(\mathbf{y})$;
2. pairwise compensability of criteria: if $\mathbf{x}, \mathbf{y} \in X$ and $v_k(\mathbf{x}) = v_k(\mathbf{y})$ $k = 1, 2$, then $\varphi(\mathbf{x}) = \varphi(\mathbf{y})$, where $v_k(\mathbf{x}) = |\{i : x_i = k\}|$ is the number of estimates of the gradation k (cardinality of the set of k gradation) in the alternative $\mathbf{x}$, $k = 1, 2, 3$;
3. threshold noncompensability: $\varphi(\underbrace{2, \ldots, 2}_{n}) > \varphi(\mathbf{x})$ $\forall \mathbf{x} \in X$: $\exists s : \ x_s = 1$;
4. the reduction axiom: if $\forall \mathbf{x}, \mathbf{y} \in X \ \exists s : \ x_s = y_s$, then $\varphi_n(\mathbf{x}) > \varphi_n(\mathbf{y}) \Leftrightarrow \varphi_{n-1}(\mathbf{x}_{-s}) > \varphi_{n-1}(\mathbf{y}_{-s})$, where $\mathbf{x}_{-s} = (x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_n)$.

It has been shown [2] that aggregation φ comes down to the application of a lexicographic rule: $\varphi(\mathbf{x}) > \varphi(\mathbf{y}) \Leftrightarrow v_1(\mathbf{x}) < v_1(\mathbf{y})$ or $\exists j \in \{1, 2\} : v_k(\mathbf{x}) = v_k(\mathbf{y})$ $\forall k \leq j$ and $v_{k+1}(\mathbf{x}) < v_{k+1}(\mathbf{y})$. A generalization of this problem to the case of m-gradation scales ($m \geq 3$) was considered in [1].

3 The procedure for Finding the Fuzzy Cardinality of a Set of Three-Grade Fuzzy Estimates

Let the alternatives be represented by n-dimensional vectors $\widetilde{\mathbf{x}} = (\widetilde{x_1}, \ldots, \widetilde{x_n})$ of fuzzy sets. Each fuzzy set $\widetilde{x_i}$ is defined on a three-gradation base set $\{L, M, H\}$ and has a membership function $\mu_{\widetilde{x_i}}$. We will represent a fuzzy set $\widetilde{x_i}$ as a vector of values of its membership function: $\widetilde{x_i} = (\mu_{\widetilde{x_i}}(L), \mu_{\widetilde{x_i}}(M), \mu_{\widetilde{x_i}}(H))$. According to the meaning of the problem, we can (and will) assume that either the vector $\widetilde{x_i}$ is strictly monotonic, or $\mu_{\widetilde{x_i}}(M) > \max\{\mu_{\widetilde{x_i}}(L), \mu_{\widetilde{x_i}}(H)\}$.

We will say that the fuzzy set $\widetilde{x_i}$ belongs to one of the classes: class L (low grades), class M (medium grades) or class H (high grades): $\widetilde{x_i} \in S \Leftrightarrow \mu_{\widetilde{x_i}}(S) = \max\limits_{Q \in \{L,M,H\}} \mu_{\widetilde{x_i}}(Q)$, $S \in \{L, M, H\}$. Let $S_{\widetilde{\mathbf{x}}} = \{\widetilde{x_i} : \widetilde{x_i} \in S\}$, $S \in \{L, M, H\}$.

We will introduce the concept of fuzzy cardinality $\widetilde{v_S}(\widetilde{\mathbf{x}})$ of estimates for each vector $\widetilde{\mathbf{x}} = (\widetilde{x_1}, \ldots, \widetilde{x_n})$ of fuzzy estimates and each class $S \in \{L, M, H\}$. Fuzzy cardinality will be defined on the base set $\{0, \ldots, n\}$ (n is the number of criteria, assessments, etc., the dimension of the vector of alternatives). The fuzzy cardinality membership function $\mu_{\widetilde{v_S}(\widetilde{\mathbf{x}})}$ of the class $S \in \{L, M, H\}$ must satisfy the following conditions.

1) $\mu_{\widetilde{v_S}(\widetilde{\mathbf{x}})}(k) = 1 \Leftrightarrow k = \left\lfloor \sum_{\widetilde{x} \in S_{\widetilde{\mathbf{x}}}} \mu_{\widetilde{x}}(S) \right\rfloor$, where $\lfloor\ \rfloor$ is rounding down. The value $\left\lfloor \sum_{\widetilde{x} \in S_{\widetilde{\mathbf{x}}}} \mu_{\widetilde{x}}(S) \right\rfloor$ is the lower estimate of the cardinality of the largest values of the membership function of all class estimates (gradation) S.

2) $\mu_{\widetilde{v_S}(\widetilde{\mathbf{x}})}(k) = 0$, if $k < \left\lfloor \sum_{\widetilde{x} \in S_{\widetilde{\mathbf{x}}}} \mu_{\widetilde{x}}(S) \right\rfloor$.

Condition 2) means that the cardinality of the set of estimates for a class S cannot be less than the number of estimates that obviously belong to this class.

Desirable properties of fuzzy cardinality would also be the following. Let $Fuz(\widetilde{\mathbf{x}}) = (Fuz(\widetilde{x_1}), \ldots, Fuz(\widetilde{x_n}))$, where Fuz is a certain degree of fuzziness of the set. For $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ vectors, comparison $\mathbf{a} \geq \mathbf{b}$ means that $a_1 \geq b_1, \ldots, a_n \geq b_n$.

3) if $Fuz(\widetilde{\mathbf{x}}) \geq Fuz(\widetilde{\mathbf{y}})$, then $Fuz(\widetilde{v_S}(\widetilde{\mathbf{x}})) \geq Fuz(\widetilde{v_S}(\widetilde{\mathbf{y}}))$ $\forall S \in \{L, M, H\}$.

4) if $Fuz(\widetilde{\mathbf{x}}) = \mathbf{0}$, then $\widetilde{v_S}(\widetilde{\mathbf{x}}) = v_S(\mathbf{x})$ $\forall S \in \{L, M, H\}$.

Due to the specified restrictions on fuzzy estimates $\widetilde{x_i}$, the last condition means that if all fuzzy estimates are non-fuzzy (i. e. $\mu_{\widetilde{x_i}}(S) \in \{0,1\}$ $\forall S \in \{L, M, H\}$), then the fuzzy cardinality of the vector estimate will coincide with the usual cardinality. In this case, the fuzzy cardinality membership function will be binary: $\mu_{\widetilde{v_S}(\widetilde{\mathbf{x}})}(k) = \begin{cases} 1, & k = |S_{\widetilde{\mathbf{x}}}|, \\ 0, & k \neq |S_{\widetilde{\mathbf{x}}}|, \end{cases}$ $k = 0, \ldots, n$. We will find the remaining values $\mu_{\widetilde{v_S}(\widetilde{\mathbf{x}})}(k)$ for $k > \left\lfloor \sum_{\widetilde{x} \in S_{\widetilde{\mathbf{x}}}} \mu_{\widetilde{x}}(S) \right\rfloor$ using the following threshold rule.

Let $S_1, S_2, S_3 \in \{L, M, H\}$ be three different classes of estimates such that $\mu_{\widetilde{x_i}}(S_1) > \mu_{\widetilde{x_i}}(S_2) \geq \mu_{\widetilde{x_i}}(S_3)$. Then we will call the estimate $\widetilde{x_i}$ the first level estimate for the class S_1. If $\mu_{\widetilde{x_i}}(S_2) > \mu_{\widetilde{x_i}}(S_3)$ is true, then the assessment $\widetilde{x_i}$ will be called the assessment of the second level for the class S_2 and the third level for the class S_3. If $\mu_{\widetilde{x_i}}(S_2) = \mu_{\widetilde{x_i}}(S_3)$ is true, then we call the estimate $\widetilde{x_i}$

an estimate of the second level for both the class S_2 and the class S_3. We will order all values $\mu_{\widetilde{x_i}}(S) = q_i$, $i = 1, \ldots, n$ in ascending order of level numbers for a fixed class S: $q_{i_1}^{(1)}, \ldots, q_{i_k}^{(1)}, q_{i_k+1}^{(2)}, \ldots, q_{i_r}^{(2)}, q_{i_r+1}^{(3)}, \ldots, q_{i_n}^{(3)}$ (the superscript is the level number).

Then we get for the first level values, according to condition 1): $\mu_{\widetilde{v_S}(\widetilde{\mathbf{x}})}(p_1) = 1$, where $p_1 := \left\lfloor q_{i_1}^{(1)} + \ldots + q_{i_k}^{(1)} \right\rfloor$. If there are no first-level estimates, then we assume $p_1 = 0$.

Next, we will consider the values of the 2^{nd} level. If there are quite a lot of them and they have large values, then this means that the values of the fuzzy cardinality membership function will be quite large for cardinalities greater than p_1. For example, the following threshold procedure may be proposed.

If $p_2 = \left\lfloor \left\{ q_{i_1}^{(1)} + \ldots + q_{i_k}^{(1)} \right\} + q_{i_k+1}^{(2)} + \ldots + q_{i_r}^{(2)} \right\rfloor \geq 1$, then $\mu_{\widetilde{v_S}(\widetilde{\mathbf{x}})}(p_1 + 1) = \ldots = \mu_{\widetilde{v_S}(\widetilde{\mathbf{x}})}(p_1 + p_2) = m_2$, where $m_2 \in (0, 1)$. Here $\{\ \}$ is the fractional part of the number. Note that $p_1 + p_2 < n$.

The values of the 3^{rd} level are taken into account in the same way. If $p_3 = \left\lfloor \left\{ \left\{ q_{i_1}^{(1)} + \ldots + q_{i_k}^{(1)} \right\} + q_{i_k+1}^{(2)} + \ldots + q_{i_r}^{(2)} \right\} + q_{i_r+1}^{(3)} + \ldots + q_{i_n}^{(3)} \right\rfloor \geq 1$, then we will increase by m_3 the membership function $\mu_{\widetilde{v_S}(\widetilde{\mathbf{x}})}$ for the values of the argument $p_1 + 1, \ldots, p_1 + p_3$, where $0 < m_3 < \min\{m_2, 1 - m_2\}$. Note that $p_1 + p_3 < n$.

Example 1. Let the vector $\widetilde{\mathbf{x}} = (\widetilde{x_1}, \ldots, \widetilde{x_5})$ from 5 fuzzy estimates be given, where $\widetilde{x_i} = (\mu_{\widetilde{x_i}}(L), \mu_{\widetilde{x_i}}(M), \mu_{\widetilde{x_i}}(H))$, $i = 1, \ldots, 5$ and $\widetilde{x_1} = (0.5,\ 0.6,\ 1)$, $\widetilde{x_2} = (0.3,\ 0.5,\ 1)$, $\widetilde{x_3} = (0.5,\ 1,\ 0.4)$, $\widetilde{x_4} = (1,\ 0.8,\ 0.3)$, $\widetilde{x_5} = (1,\ 0.5,\ 0.2)$. Then we will get the following results of calculating the values of the fuzzy cardinality membership function for each class (which we will also write in the form of vectors) $\widetilde{v_S}(\widetilde{\mathbf{x}}) = \left(\mu_{\widetilde{v_S}(\widetilde{\mathbf{x}})}(0), \ldots, \mu_{\widetilde{v_S}(\widetilde{\mathbf{x}})}(5)\right)$, $S \in \{L, M, H\}$:

a) the vector of low grades ordered by level is equal to $\Big(1^{(1)}; 1^{(1)}; 0.5^{(2)}; 0.5^{(3)}; 0.3^{(3)}\Big)$. Therefore, we have $p_1 = 1^{(1)} + 1^{(1)} = 2$, $p_2 = \left\lfloor 0.5^{(2)} \right\rfloor = 0$, $p_3 = \left\lfloor \left\{ 0.5^{(2)} \right\} + 0.5^{(3)} + 0.3^{(3)} \right\rfloor = 1$. Hence, $\widetilde{v_L}(\widetilde{\mathbf{x}}) = (0, 0, 1, m_3, 0, 0)$.

b) the vector of level-ordered values for medium scores is equal to $\Big(1^{(1)}; 0.8^{(2)}; 0.6^{(2)}; 0.5^{(2)}; 0.5^{(2)}\Big)$. Therefore, we have $p_1 = 1^{(1)} = 1$, $p_2 = \Big\lfloor 0.8^{(2)} + 0.6^{(2)} + 0.5^{(2)} + 0.5^{(2)} \Big\rfloor = 2$, $p_3 = 0$ and $\widetilde{v_M}(\widetilde{\mathbf{x}}) = (0, 1, m_2, m_2, 0, 0)$.

c) the vector of level-ordered values for high scores is equal to $\big(1^{(1)}; 1^{(1)}; 0.4^{(3)}; 0.3^{(3)}; 0.2^{(3)}\big)$. Therefore, we have $p_1 = 1^{(1)} + 1^{(1)} = 2$, $p_2 = 0$, $p_3 = \left\lfloor 0.4^{(3)} + 0.3^{(3)} + 0.2^{(3)} \right\rfloor = 0$ and $\widetilde{v_H}(\widetilde{\mathbf{x}}) = (0, 0, 1, 0, 0, 0)$.

Here m_2, m_3 are some threshold values that satisfy the conditions $m_2 \in (0, 1)$, $0 < m_3 < \min\{m_2, 1 - m_2\}$.

4 Comparison of Fuzzy Cardinality and Ranking of Alternatives

Let $\mathcal{V}_n$ be the set of all fuzzy cardinalities for n fuzzy estimates.

To apply the lexicographic rule for ranking a set of vector fuzzy alternatives $\{\widetilde{\mathbf{x}}\}$ with respect to fuzzy cardinality $\widetilde{\mathbf{v}}(\widetilde{\mathbf{x}}) = (\widetilde{v_L}(\widetilde{\mathbf{x}}), \widetilde{v_M}(\widetilde{\mathbf{x}}), \widetilde{v_H}(\widetilde{\mathbf{x}}))$, it is necessary to use some rule for ordering fuzzy sets. This can be done using some defuzzification function $F : \mathcal{V}_n \to \mathbb{R}$.

We will assume that the fuzzy cardinalities of class $S \in \{L, M, H\}$ estimates are in the relation $\widetilde{v_S}(\widetilde{\mathbf{x}}) \prec \widetilde{v_S}(\widetilde{\mathbf{y}})$ for two alternatives $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{y}}$ if $F(\widetilde{v_S}(\widetilde{\mathbf{x}})) < F(\widetilde{v_S}(\widetilde{\mathbf{y}}))$ and are equal $\widetilde{v_S}(\widetilde{\mathbf{x}}) \sim \widetilde{v_S}(\widetilde{\mathbf{y}})$ if $F(\widetilde{v_S}(\widetilde{\mathbf{x}})) = F(\widetilde{v_S}(\widetilde{\mathbf{y}}))$.

For example, if we use center of gravity

$$G(\widetilde{v_S}(\widetilde{\mathbf{x}})) = \sum_{i=0}^{n} i\mu_{\widetilde{v_S}(\widetilde{\mathbf{x}})}(i) \Bigg/ \sum_{i=0}^{n} \mu_{\widetilde{v_S}(\widetilde{\mathbf{x}})}(i)$$

as the defuzzification function, we get for the example above:

$$G(\widetilde{v_L}(\widetilde{\mathbf{x}})) = \frac{2+3m_3}{1+m_3}, \;\; G(\widetilde{v_M}(\widetilde{\mathbf{x}})) = \frac{1+5m_2}{1+2m_2}, \;\; G(\widetilde{v_H}(\widetilde{\mathbf{x}})) = 2.$$

For any acceptable threshold values m_2 and m_3, we have $G(\widetilde{v_M}(\widetilde{\mathbf{x}})) < G(\widetilde{v_H}(\widetilde{\mathbf{x}})) < G(\widetilde{v_L}(\widetilde{\mathbf{x}}))$. Therefore, the ranking $\widetilde{v_M}(\widetilde{\mathbf{x}}) \prec \widetilde{v_H}(\widetilde{\mathbf{x}}) \prec \widetilde{v_L}(\widetilde{\mathbf{x}})$ is correct.

Another way to compare the cardinality of sets of fuzzy estimates is to use the same lexicographic rule. Let $\widetilde{v_S}(\widetilde{\mathbf{x}}) = (a_0, \ldots, a_n)$, $\widetilde{v_S}(\widetilde{\mathbf{y}}) = (b_0, \ldots, b_n)$. Then we will assume that $\widetilde{v_S}(\widetilde{\mathbf{x}}) \prec \widetilde{v_S}(\widetilde{\mathbf{y}})$ if $a_0 < b_0$ or $\exists k \in \{0, \ldots, n-1\}$: $a_0 = b_0, \ldots, a_k = b_k, a_{k+1} < b_{k+1}$. Otherwise, we assume that $\widetilde{v_S}(\widetilde{\mathbf{x}}) \sim \widetilde{v_S}(\widetilde{\mathbf{y}})$.

Now, the threshold aggregation rule for two alternatives $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{y}}$ will be as follows:

$$\begin{aligned}\varphi(\widetilde{\mathbf{x}}) > \varphi(\widetilde{\mathbf{y}}) \Leftrightarrow \;& \widetilde{v_L}(\widetilde{\mathbf{x}}) \prec \widetilde{v_L}(\widetilde{\mathbf{y}}) \;\text{ or }\; \widetilde{v_L}(\widetilde{\mathbf{x}}) \sim \widetilde{v_L}(\widetilde{\mathbf{y}}), \;\; \widetilde{v_M}(\widetilde{\mathbf{x}}) \prec \widetilde{v_M}(\widetilde{\mathbf{y}}) \\ &\text{or }\; \widetilde{v_L}(\widetilde{\mathbf{x}}) \sim \widetilde{v_L}(\widetilde{\mathbf{y}}), \;\; \widetilde{v_M}(\widetilde{\mathbf{x}}) \sim \widetilde{v_M}(\widetilde{\mathbf{y}}), \; \widetilde{v_H}(\widetilde{\mathbf{x}}) \prec \widetilde{v_H}(\widetilde{\mathbf{y}}).\end{aligned}$$

5 Numerical Example

Let us illustrate the proposed fuzzy threshold aggregation procedure using the example of ranking reviews of articles/conference reports in the EasyChair conference management system (`https://easychair.org`). The septenary grading system $z_i \in \{-3, -2, -1, 0, 1, 2, 3\}$ is used in the EasyChair system. The system ranks correspond to the recommendations "strong reject", "reject", "weak reject", "borderline paper", "weak accept", "accept", "strong accept". In addition, the reviewer gives a rating on a five-star scale (0.2 – "none", 0.4 – "low", 0.6 – "medium", 0.8 – "high", 1 – "expert") about the degree of confidence in the correctness of his decision: $\lambda_i \in \{0.2, 0.4, 0.6, 0.8, 1\}$. This example was considered in [5] to study the metric approach in the fuzzy threshold aggregation problem.

In addition, this example was also considered in [6] to illustrate the evidential aggregation and ranking of expert ordinal data.

Point data $\left\{(z_i^{(k)}, \lambda_i^{(k)})\right\}_{i=1}^{5}$ of $n = 5$ reviewers regarding 4 articles are presented in Table 1 (i is the reviewer's index, k is the article's index, $k = 1, \dots, 4$).

Table 1. Initial data on reviewers' assessments

	paper 1	paper 2	paper 3	paper 4
reviewer 1	$(2, 0.8)$	$(2, 1)$	$(1, 0.8)$	$(0, 0.6)$
reviewer 2	$(1, 1)$	$(2, 0.8)$	$(2, 0.6)$	$(1, 0.4)$
reviewer 3	$(0, 0.8)$	$(0, 0.6)$	$(-1, 0.6)$	$(1, 1)$
reviewer 4	$(3, 0.4)$	$(-1, 0.4)$	$(0, 0.6)$	$(2, 0.2)$
reviewer 5	$(2, 0.6)$	$(1, 0.6)$	$(1, 1)$	$(2, 1)$

Let's transform each pair (z, λ) into a three-grade fuzzy estimate $\widetilde{x} = \widetilde{x}(z, \lambda) = (\mu_{\widetilde{x}}(L), \mu_{\widetilde{x}}(M), \mu_{\widetilde{x}}(H))$ using some rule for blurring point estimates. Various blur rules are widely used in theory and applications. For example, the optimization procedure for converting point expert estimates into interval ones was considered in [4]. The procedure for converting interval expert data into fuzzy data, taking into account additional information about the decision maker, was considered in [3]. The transformation of point data of an ordinal scale into some sets of ranks was considered in [6].

We will use the following blur rule below: if $z \in H = \{1, 2, 3\}$ (high estimates), then $\widetilde{x} = \left(\frac{\lambda}{2z+\lambda}, \frac{\lambda}{z+\lambda}, \lambda\right)$; if $z \in L = \{-3, -2, -1\}$ (low estimates), then $\widetilde{x} = \left(\lambda, \frac{\lambda}{|z|+\lambda}, \frac{\lambda}{2|z|+\lambda}\right)$; if $z \in M = \{0\}$ (medium estimate), then $\widetilde{x} = \left(\frac{\lambda}{1+2\lambda}, \lambda, \frac{\lambda}{1+2\lambda}\right)$.

Remark 1. In general, we will require the transformation $\widetilde{x}(z, \lambda)$ to satisfy the following conditions:

1. $\mu_{\widetilde{x}(z,\lambda)}(S) \leq \lambda \ \forall S \in \{L, M, H\}$, $\forall (z, \lambda)$ and $\mu_{\widetilde{x}(z,\lambda)}(S) = \lambda \Leftrightarrow z \in S$.
2. values $\mu_{\widetilde{x}(z,\lambda)}(L)$, $\mu_{\widetilde{x}(z,\lambda)}(M)$, $\mu_{\widetilde{x}(z,\lambda)}(H)$ must satisfy the above monotonicity conditions for different z;
3. if $\lambda' \geq \lambda''$, then $Fuz\left(\widetilde{x}(z, \lambda')\right) \leq Fuz\left(\widetilde{x}(z, \lambda'')\right)$;
   if $|z'| \geq |z''|$, then $Fuz\left(\widetilde{x}(z', \lambda)\right) \leq Fuz\left(\widetilde{x}(z'', \lambda)\right)$.

The transformations used satisfy these requirements.

The fuzzy estimates obtained in this way are presented in Table 2.

Using the method described above, we will find the fuzzy cardinality of all reviewer ratings for all classes and for all articles.

Table 2. Fuzzy assessments of reviewers, their fuzzy cardinality, centers of gravity of fuzzy cardinality and cardinality of non-fuzzy assessments

	paper 1	paper 2	paper 3	paper 4
rev. 1	$\left(\frac{1}{6}, \frac{2}{7}, \frac{4}{5}\right)$	$\left(\frac{1}{5}, \frac{1}{3}, 1\right)$	$\left(\frac{2}{7}, \frac{4}{9}, \frac{4}{5}\right)$	$\left(\frac{3}{11}, \frac{3}{5}, \frac{3}{11}\right)$
rev. 2	$\left(\frac{1}{3}, \frac{1}{2}, 1\right)$	$\left(\frac{1}{6}, \frac{2}{7}, \frac{4}{5}\right)$	$\left(\frac{3}{23}, \frac{3}{13}, \frac{3}{5}\right)$	$\left(\frac{1}{6}, \frac{2}{7}, \frac{2}{5}\right)$
rev. 3	$\left(\frac{4}{13}, \frac{4}{5}, \frac{4}{13}\right)$	$\left(\frac{3}{11}, \frac{3}{5}, \frac{3}{11}\right)$	$\left(\frac{3}{5}, \frac{3}{8}, \frac{3}{13}\right)$	$\left(\frac{1}{3}, \frac{1}{2}, 1\right)$
rev. 4	$\left(\frac{1}{16}, \frac{2}{17}, \frac{2}{5}\right)$	$\left(\frac{2}{5}, \frac{2}{7}, \frac{1}{6}\right)$	$\left(\frac{3}{11}, \frac{3}{5}, \frac{3}{11}\right)$	$\left(\frac{1}{21}, \frac{1}{11}, \frac{1}{5}\right)$
rev. 5	$\left(\frac{3}{23}, \frac{3}{13}, \frac{3}{5}\right)$	$\left(\frac{3}{13}, \frac{3}{8}, \frac{3}{5}\right)$	$\left(\frac{1}{3}, \frac{1}{2}, 1\right)$	$\left(\frac{1}{5}, \frac{1}{3}, 1\right)$
$\widetilde{v_L}^{(k)}$	$(1, m_3, 0, 0, 0, 0)$	$(1, m_3, 0, 0, 0, 0)$	$(1, m_3, 0, 0, 0, 0)$	$(1, m_3, 0, 0, 0, 0)$
$\widetilde{v_M}^{(k)}$	$(1, m_2, 0, 0, 0, 0)$	$(1, m_3, 0, 0, 0, 0)$	$(1, m_2, m_2, 0, 0, 0)$	$(1, m_2, 0, 0, 0, 0)$
$\widetilde{v_H}^{(k)}$	$(0, 0, 1, m_2, 0, 0)$	$(0, 0, 1, 0, 0, 0)$	$(0, 0, 1, 0, 0, 0)$	$(0, 0, 1, 0, 0, 0)$
$\mathbf{G}^{(k)}$	$\left(\frac{m_3}{1+m_3}, \frac{m_2}{1+m_2}, \frac{2+3m_2}{1+m_2}\right)$	$\left(\frac{m_3}{1+m_3}, \frac{m_3}{1+m_3}, 2\right)$	$\left(\frac{m_3}{1+m_3}, \frac{3m_2}{1+2m_2}, 2\right)$	$\left(\frac{m_3}{1+m_3}, \frac{m_2}{1+m_2}, 2\right)$
$\mathbf{v}^{(k)}$	$(0, 1, 4)$	$(1, 1, 3)$	$(1, 1, 3)$	$(0, 1, 4)$

The corresponding results are shown in Table 2 in vectors $\widetilde{v_S}^{(k)} = \widetilde{v_S}\left(\widetilde{\mathbf{x}}^{(k)}\right)$, $k = 1, \ldots, 4$, $S \in \{L, M, H\}$. Let's find the vectors of the centers of gravity of the fuzzy cardinalities $\mathbf{G}^{(k)} = \left(G\left(\widetilde{v_L}^{(k)}\right), G\left(\widetilde{v_M}^{(k)}\right), G\left(\widetilde{v_H}^{(k)}\right)\right)$ of all articles (see Table 2).

We will obtain the following ranking of alternatives (articles) after lexicographic comparison of vectors $\mathbf{G}^{(k)}$, $k = 1, \ldots, 4$ taking into account restrictions on threshold values $m_2 \in (0, 1)$, $0 < m_3 < \min\{m_2, 1 - m_2\}$: $\varphi\left(\widetilde{\mathbf{x}}^{(2)}\right) > \varphi\left(\widetilde{\mathbf{x}}^{(4)}\right) > \varphi\left(\widetilde{\mathbf{x}}^{(1)}\right) > \varphi\left(\widetilde{\mathbf{x}}^{(3)}\right)$.

The same ranking will be obtained if the cardinalities of gradations are compared lexicographically.

Note that if we take into account only the three-grade recommendations of reviewers ($L = \{-3, -2, -1\}$, $M = \{0\}$, $H = \{1, 2, 3\}$) and do not take into account the degree of confidence, we obtain the following vectors of cardinality of assessments $\mathbf{v}^{(k)} = \left(v_L(\mathbf{x}^{(k)}), v_M(\mathbf{x}^{(k)}), v_H(\mathbf{x}^{(k)})\right)$ for each k-article, $k = 1, \ldots, 4$ (see the last line in Table 2). Then the ranking of these articles will be as follows: $\varphi(\mathbf{x}^{(1)}) = \varphi(\mathbf{x}^{(4)}) > \varphi(\mathbf{x}^{(2)}) = \varphi(\mathbf{x}^{(3)})$. Approximately the same ranking will be obtained using the metric approach described in [5]. The main difference in the results of the new and non-fuzzy (as well as metric) approaches is the rearrangement of alternatives $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$. The alternative $\mathbf{x}^{(1)}$ will be better than the $\mathbf{x}^{(2)}$ under non-blurred threshold aggregation because the $\mathbf{x}^{(2)}$ has one low score, while the $\mathbf{x}^{(1)}$ has no low scores.

But a low score in the alternative $\mathbf{x}^{(2)}$ has a low degree of confidence. Therefore, it has little effect on the cardinality of low estimates under fuzzy threshold aggregation.

6 Conclusion

A new approach to threshold aggregation and ranking of vector alternatives specified by fuzzy evaluations of criteria on a three-graded base set is proposed in this article. This approach is based on calculating and comparing the fuzzy cardinality of sets of estimates for each gradation for all criteria and for each alternative. The general properties that the fuzzy cardinality of a set of fuzzy estimators must satisfy are discussed. The threshold procedure for constructing the fuzzy cardinality of a set of fuzzy estimates is considered.

The proposed method is illustrated by the example of ranking articles based on their assessments by reviewers and taking into account information about the degree of confidence of reviewers in their assessments. This assessment format is currently implemented in many conference management systems. For example, in EasyChair. Data from reviewers on different articles in the form of pairs "evaluation – degree of confidence" are converted into fuzzy three-gradation sets, which are then ranked according to the developed methodology. It is shown that the result of such ranking can differ significantly from non-blurred threshold ranking.

A general scheme for ranking fuzzy vector alternatives based on calculating the fuzzy cardinality of a set of estimates is proposed in the article. This scheme can be refined and upgraded for a specific application. For example, the values of measures of consistency of fuzzy estimates for each alternative can be taken into account in the comparison. Changes in the values of the membership function in the procedure for generating fuzzy cardinality can be different coordinatewise, etc.

The development of axiomatics for fuzzy threshold aggregation is one of the possible directions for future research.

References

1. Aleskerov, F., Chistyakov, V., Kalyagin, V.: Social threshold aggregations. Social Choice Welfare **35**, 627–646 (2010)
2. Aleskerov, F.T., Yakuba, V.I.: A Method for Threshold Aggregation of Three-Grade Rankings. Doklady Mathematics **75**(2), 322–324 (2007)
3. Lepskiy, A.: Conflict Measure of Belief Functions with Blurred Focal Elements on the Real Line. In: Denœux, T., Lefèvre, E., Liu, Z., Pichon, F. (eds.) Belief Functions: Theory and Applications. BELIEF 2021. LNCS, vol.12915, pp. 197–206. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-88601-1
4. Lepskiy, A.: On Optimal Blurring of Point Expert Estimates and their Aggregation in the Framework of Evidence Theory. Procedia Computer Science **214**, 573–580 (2022)
5. Lepskiy, A.: Fuzzy Threshold Aggregation. In: Kahraman, C., Sari, I.U., Oztaysi, B., Cebi, S., Cevik Onar, S., Tolga, A.Ç. (eds.) Intelligent and Fuzzy Systems. INFUS 2023. LNNS, vol. 758, pp. 69–76. Springer, Cham (2023). https://doi.org/10.1007/978-3-031-39774-5
6. Lepskiy, A.: Evidence-Based Aggregation and Ranking in an Ordinal Scale. Procedia Computer Science **221C**, 1066–1073 (2023)