Fuzzy Threshold Aggregation ⋆

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Abstract. The threshold aggregation rule used to rank alternatives that are evaluated against a set of criteria is known in decision theory. The generalization of the threshold aggregation rule to the case when the estimates in the alternatives are described by fuzzy numbers is considered in the paper. The fuzzy threshold aggregation procedure has been developed and studied. The procedure for blurring point data by information about the reliability of such data has been developed and studied as well. An example of using fuzzy threshold aggregation when making decisions on the admission of articles in conference management systems is considered (for example, EasyChair).

Keywords: Threshold rule · Aggregation of alternatives · Fuzzy criteria · Conference management systems.

1 Introduction

The rules for aggregation of individual preferences are considered and investigated in the social choice theory [1]. Such rules are used in the problem of ranking alternatives, each of which is evaluated according to a certain set of criteria. In some cases, aggregation rules should be non-compensatory. This implies that low scores on one criterion cannot be compensated for by high scores on others. For example, such a rule is often used when making decisions about the publication of articles based on the feedback of several reviewers, when choosing products by characteristics, etc. The so-called threshold rule [2,3] is one of the popular aggregation rules that has a non-compensatory property.

In some cases, some or all of the characteristics of alternatives may be fuzzy. Then the problem of generalizing the threshold aggregation rule to the case of fuzzy data is relevant. The present article is devoted to the solution of this problem.

The application of the proposed generalization of the threshold aggregation rule to fuzzy data is illustrated by the example of ranking articles according

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to the results of (point) assessments of reviewers and considering the degrees of confidence in their assessments. These degrees of confidence are used to blur point estimates and generate fuzzy data. The general properties of such blurs are also discussed in the article.

The rest of the article has the following structure. The axiomatics of the non-fuzzy threshold aggregation rule is discussed in Section 2. In Section 3, the problem of threshold aggregation with fuzzy data is formulated and a general approach to solving this problem is discussed. The formation of fuzzy estimates from point expert data and information about their reliability is discussed in Section 4. Section 5 gives a numerical example of applying the threshold aggregation rule with fuzzy data when reviewing articles in a conference management system. Finally, Section 6 draws some conclusions from the study.

2 Non-Fuzzy Formulation of the Threshold Aggregation Problem

The problem of ranking alternatives of a certain set X of evaluated by n criteria in a three-gradation scale is being considered. We can assume in this case that the alternatives are represented by n-dimensional vectors: $\mathbf{x} = (x_1, ..., x_n)$, where $x_i \in \{1, 2, 3\}$. It is required to find such a transformation (aggregation operator) $\varphi_n = \varphi : X \to \mathbb{R}$ that satisfies the conditions (axioms) [3]:

1) Pareto-domination: if $\mathbf{x}, \mathbf{y} \in X$ and $x_i \geq y_i \quad \forall i,\ \exists s :\ x_s > y_s$, then $\varphi(\mathbf{x}) > \varphi(\mathbf{y})$;
2) pairwise compensability of criteria: if $\mathbf{x}, \mathbf{y} \in X$ and $v_k(\mathbf{x}) = v_k(\mathbf{y})\ k = 1, 2$, then $\varphi(\mathbf{x}) = \varphi(\mathbf{y})$, where $v_k(\mathbf{x}) = |\{i : x_i = k\}|$ is the number of estimates of k in the alternative $\mathbf{x}$, $k = 1, 2, 3$;
3) threshold noncompensability: $\varphi(\underbrace{2, \ldots, 2}_{n}) > \varphi(\mathbf{x})\ \forall \mathbf{x} \in X$: $\exists s :\ x_s = 1$;
4) the reduction axiom: if $\forall \mathbf{x}, \mathbf{y} \in X\ \exists s :\ \ x_s = y_s$, then $\varphi_n(\mathbf{x}) > \varphi_n(\mathbf{y}) \Leftrightarrow \varphi_{n-1}(\mathbf{x}_{-s}) > \varphi_{n-1}(\mathbf{y}_{-s})$, where $\mathbf{x}_{-s} = (x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_n)$.

This problem was formulated and studied in [3]. It is shown that the lexicographic aggregation rule is a solution to this problem: $\varphi(\mathbf{x}) > \varphi(\mathbf{y}) \Leftrightarrow v_1(\mathbf{x}) < v_1(\mathbf{y})$ or $\exists j \in \{1, 2\} : v_k(\mathbf{x}) = v_k(\mathbf{y})\ \forall k \leq j$ and $v_{k+1}(\mathbf{x}) < v_{k+1}(\mathbf{y})$. This problem was generalized in [2] to the case of m-gradation scales, $m \geq 3$.

3 Formulation and Solution of the Problem of Threshold Aggregation with Fuzzy Data

Let us now assume that the alternatives are represented by n-dimensional vectors of fuzzy numbers $\widetilde{\mathbf{x}} = (\widetilde{x_1}, \ldots, \widetilde{x_n})$. Each fuzzy number belongs to one of three classes: the low score class L, the median score class M, or the high score class H. The distribution of fuzzy numbers by class can either be known in advance, or can be determined by the nearest neighbour method with respect to reference numbers L_0, M_0 and H_0, and each class $\widetilde{x_i} \in \underset{S\in\{L,M,H\}}{\arg\min}\ d(\widetilde{x_i}, S_0)$, where d is some metric (or pseudometric) on the set of fuzzy numbers [5].

We will assume that the supports of all fuzzy estimators of the class L are located on the segment $[-a, 0]$, $a > 0$, the supports of all fuzzy estimators of the class H are located on the segment $[0, a]$, $a > 0$, and the supports of all fuzzy estimators of the class M are located on the segment $[-b, b]$, $0 < b < a$. Crisp numbers $L_0 = -a$, $M_0 = 0$ and $H_0 = a$ are reference elements.

We will consider a set of median positive estimates $M^+ = \{\widetilde{x} \in M : d(\widetilde{x}, M_0^+) \le d(\widetilde{x}, M_0^-)\}$ and a set of median negative estimates $M^- = \{\widetilde{x} \in M : d(\widetilde{x}, M_0^-) \le d(\widetilde{x}, M_0^+)\}$, which are subsets of the set of median estimates M, where $M_0^- = -b$, $M_0^+ = b$ are the reference estimates of subclasses M^- and M^+, respectively.

Note that those estimates for which equality $d(\widetilde{x}, M_0^-) = d(\widetilde{x}, M_0^+)$ holds (and only they) fall into both subsets.

General scheme of threshold fuzzy ranking.

The following steps are performed for each alternative $\widetilde{\mathbf{x}} = (\widetilde{x_1}, \ldots, \widetilde{x_n})$ and each class $S \in \{L, M^-, M^+, H\}$.

1. The distances $d\left(\widetilde{x_i}, S_0\right)$ between all estimates $\widetilde{x_i} \in S$ of one class and the reference fuzzy (or crisp) number S_0 of this class are determined (the values $d\left(\widetilde{x_i}, M_0\right)$ and $d\left(\widetilde{x_i}, M_0^{\pm}\right)$ are calculated for the class M).

2. The normalized function $F_S\left(\widetilde{x_i}\right) = \psi\left(d\left(\widetilde{x_i}, S_0\right)\right)$ of the proximity of the estimate to the reference number of the class is calculated, where nonincreasing function $\psi : [0, +\infty) \to [0, 1]$ satisfies the condition $\psi(0) = 1$. The value $F_S\left(\widetilde{x_i}\right)$ characterizes the normalized degree of confidence that the estimate $\widetilde{x_i} \in S$. It can be considered as a function of belonging to a subset S defined on a set of fuzzy numbers.

3. Let's find values

$$v_S\left(\widetilde{\mathbf{x}}\right) = \sum_{\widetilde{x_i} \in S} F_S\left(\widetilde{x_i}\right). \tag{1}$$

for the alternative $\widetilde{\mathbf{x}} = (\widetilde{x_1}, \ldots, \widetilde{x_n})$ and each class $S \in \{L, M^-, M^+, H\}$. The value $v_S(\mathbf{x})$ characterizes the cardinality of the set of fuzzy estimates of the class S.

4. Let's apply the lexicographic aggregation rule: $\varphi\left(\widetilde{\mathbf{x}}\right) > \varphi\left(\widetilde{\mathbf{y}}\right) \Leftrightarrow v_L\left(\widetilde{\mathbf{x}}\right) < v_L\left(\widetilde{\mathbf{y}}\right)$ or $v_L\left(\widetilde{\mathbf{x}}\right) = v_L\left(\widetilde{\mathbf{y}}\right)$, $v_{M^-}\left(\widetilde{\mathbf{x}}\right) < v_{M^-}\left(\widetilde{\mathbf{y}}\right)$ or $v_L\left(\widetilde{\mathbf{x}}\right) = v_L\left(\widetilde{\mathbf{y}}\right)$, $v_{M^-}\left(\widetilde{\mathbf{x}}\right) = v_{M^-}\left(\widetilde{\mathbf{y}}\right)$, $v_{M^+}\left(\widetilde{\mathbf{x}}\right) < v_{M^+}\left(\widetilde{\mathbf{y}}\right)$ or $v_L\left(\widetilde{\mathbf{x}}\right) = v_L\left(\widetilde{\mathbf{y}}\right)$, $v_{M^-}\left(\widetilde{\mathbf{x}}\right) = v_{M^-}\left(\widetilde{\mathbf{y}}\right)$, $v_{M^+}\left(\widetilde{\mathbf{x}}\right) = v_{M^+}\left(\widetilde{\mathbf{y}}\right)$, $v_H\left(\widetilde{\mathbf{x}}\right) < v_H\left(\widetilde{\mathbf{y}}\right)$.

Remark 1. If the fuzzy estimates are crisp numbers on a three-gradation scale $\{L_0, M_0, H_0\}$, then the values $v_S\left(\widetilde{\mathbf{x}}\right)$ coincide with the values ,$v_S\left(\mathbf{x}\right) = |\{i : x_i \in S\}|$ $S \in \{L, M^-, M^+, H\}$ $(M^- = M^+ = M)$ and the threshold fuzzy aggregation will coincide with the usual threshold aggregation.

Remark 2. We can use the robust "soft" comparison $<_\varepsilon$ and equality $=_\varepsilon$ instead of the specified "hard" comparison of the values $v_S(\widetilde{\mathbf{x}})$ in step 4 of the aggregation procedure: $a <_\varepsilon b$, if $a < b - \varepsilon$ and $a =_\varepsilon b$, if $|a - b| \leq \varepsilon$, where $\varepsilon \geq 0$ (the persistence parameter).

In this article, we will consider the value $d_\delta(A, B) = |Val(A) - Val(B)| + \delta\,|Am(A) - Am(B)|$ [7] as the distance between fuzzy numbers A and B, where $Val(A)$ and $Am(A)$ are the expected value and ambiguity of the fuzzy number A, respectively. The coefficient $\delta > 0$ regulates the priority of importance between the divergence of expected values and ambiguity. Below we will assume that $\delta \leq \frac{1}{2}$. If $A_\alpha = \{t : \mu_A(t) \geq \alpha\} = [l_A(\alpha), r_A(\alpha)]$ is a α-cut of a fuzzy number (μ_A is a membership function), $\alpha \in (0, 1]$, then [6] $Val(A) = \frac{1}{2}\int_0^1 (l_A(\alpha) + r_A(\alpha))\, d\alpha$, $Am(A) = \int_0^1 (r_A(\alpha) - l_A(\alpha))\, d\alpha$ [4]. Then the following lemma is true.

Lemma 1. *The following equalities are valid under the indicated assumptions:*

$$d_\delta(\widetilde{x}, L_0) = a + Val(\widetilde{x}) + \delta Am(\widetilde{x}) \;\; \forall \widetilde{x} \in L,$$

$$\begin{aligned} d_\delta(\widetilde{x}, M_0^-) &= b + Val(\widetilde{x}) + \delta Am(\widetilde{x}) \;\; \forall \widetilde{x} \in M, \\ d_\delta(\widetilde{x}, M_0^+) &= b - Val(\widetilde{x}) + \delta Am(\widetilde{x}) \;\; \forall \widetilde{x} \in M, \\ d_\delta(\widetilde{x}, H_0) &= a - Val(\widetilde{x}) + \delta Am(\widetilde{x}) \;\; \forall \widetilde{x} \in H. \end{aligned}$$

We will use a linear function $\psi(t) = 1 - \frac{1}{t_0} t$, $t \in [0, t_0]$ at step 2 of the threshold aggregation. Then the validity of the following assertion follows from the lemma.

Proposition 1. *The following equalities are valid under the indicated assumptions:*

$$\begin{aligned} v_L(\widetilde{\mathbf{x}}) &= \frac{\delta}{1+\delta}\,|L(\widetilde{\mathbf{x}})| - \frac{1}{a(1+\delta)} \sum_{\widetilde{x}_i \in L} (Val(\widetilde{x}_i) + \delta Am(\widetilde{x}_i)), \\ v_{M^-}(\widetilde{\mathbf{x}}) &= \frac{1+2\delta}{2(1+\delta)}\left|M^-(\widetilde{\mathbf{x}})\right| - \frac{1}{2b(1+\delta)} \sum_{\widetilde{x}_i \in M^-} (Val(\widetilde{x}_i) + \delta Am(\widetilde{x}_i)), \\ v_{M^+}(\widetilde{\mathbf{x}}) &= \frac{1+2\delta}{2(1+\delta)}\left|M^+(\widetilde{\mathbf{x}})\right| + \frac{1}{2b(1+\delta)} \sum_{\widetilde{x}_i \in M^+} (Val(\widetilde{x}_i) - \delta Am(\widetilde{x}_i)), \\ v_H(\widetilde{\mathbf{x}}) &= \frac{\delta}{1+\delta}\,|H(\widetilde{\mathbf{x}})| + \frac{1}{a(1+\delta)} \sum_{\widetilde{x}_i \in H} (Val(\widetilde{x}_i) - \delta Am(\widetilde{x}_i)), \end{aligned} \tag{2}$$

where $S(\widetilde{\mathbf{x}}) = \{\widetilde{x}_i \in S\}$, $S \in \{L, M^-, M^+, H\}$.

4 Formation of Fuzzy Estimates from Point Data

Below we will consider an example of threshold aggregation of the results of reviewing conference reports. These results from each reviewer i are represented by a pair of numbers, a point estimate of the quality of the article x_i (higher value corresponds to higher quality) and the degree of confidence in the correctness of their decision λ_i (higher value corresponds to greater confidence). Therefore, we consider the general procedure for "blurring" point estimates with respect to this information $\{(x_i, \lambda_i)\}_{i=1}^n$.

Without loss of generality, we can assume that $x_i \in [-a, a]$. Information about the degree of confidence in the correctness of one's decision can be used to blur interval or point estimates. The lower the degree of confidence, the greater should be the degree of blurring and ambiguity of fuzzy estimates. It can be considered that the degree of blur should be zero for estimates with the highest degree of confidence. In addition, "extreme" estimates with a low degree of confidence should not move away from the neutral estimate after blurring. Shifting a low confidence estimate to a neutral estimate makes it less conflicting with other estimates, which is in line with the "no conflict with ignorance" paradigm.

Formally, the dependence of blurring on the degree of confidence can be modeled using the modifier $m_\lambda : [0,1] \to [0,1]$, $m_\lambda(0) = 0$, $m_\lambda(1) = 1$ according to the rule $\mu_{m_\lambda A}(t) = m_\lambda(\mu_A(t))$, where A is a fuzzy estimate with a degree of confidence $\lambda \in [0,1]$. The parametric modifier m_λ must satisfy the conditions:

1) $m_1 A = A$;
2) $Fuz(m_\lambda A) \geq Fuz(m_\tau A)$ for $\lambda \leq \tau$, where Fuz is some (fixed) degree of fuzziness of a fuzzy number;
3) $Am(m_\lambda A) \geq Am(m_\tau A)$ for $\lambda \leq \tau$, where Am is some (fixed) measure of the ambiguity of a fuzzy number;
4) $d(m_\lambda A, M_0) \leq d(m_\tau A, M_0)$ for $\lambda \leq \tau$ and A is an "extreme" fuzzy estimate, where d is some metric (or pseudometric) on the set of fuzzy numbers, M_0 is the reference number of the median score class.

For example, if fuzzy estimates are described by trapezoidal fuzzy numbers, then we can assume that the recommendation itself determines the kernel of the corresponding fuzzy number, and the degree of confidence determines the discrepancy between the kernel and the support of the fuzzy number. The lower the degree of confidence, the larger the discrepancy should be. The kernel should match the support in the case of the highest degree of confidence. In this case, the initial estimate A will be represented by a real number or segment $[a_1, a_2]$, $a_1 \leq a_2$, which can be considered a trapezoidal fuzzy number $A = (a_1, a_1, a_2, a_2)$. The modifier must satisfy conditions 1), 4) and:

2') $\ker(m_\lambda A) \subseteq \ker(m_\tau A)$ and $\mathrm{supp}(m_\lambda A) \supseteq \mathrm{supp}(m_\tau A)$ for $\lambda \leq \tau$;
3') $|\ker(m_\lambda A)| + |\mathrm{supp}(m_\lambda A)| \geq |\ker(m_\tau A)| + |\mathrm{supp}(m_\tau A)|$ for $\lambda \leq \tau$.

Conditions 2) and 3) follow from conditions 2') and 3') if we use $Fuz(A) = |\mathrm{supp}\, A \setminus \ker A|$ [7], $Am(A) = \frac{1}{2}(|\mathrm{supp}\, A| + |\ker A|)$ measures as degrees of fuzziness and ambiguity of trapezoidal fuzzy numbers, where $|\cdot|$ is the measure of a set on the number line.

Example 1. Let nonnegative nonincreasing functions $h_1(\lambda)$ and $h_2(\lambda)$ satisfy the condition $h_1(1) = h_2(1) = 0$. Consider the following external blur modifier

$$m_\lambda A = (a_1 - h_1(\lambda), a_1, a_2, a_2 + h_2(\lambda)).$$

Then we have $Fuz\,(m_\lambda A) = h_1(\lambda) + h_2(\lambda)$ and $Am\,(m_\lambda A) = a_2 - a_1 + \frac{1}{2}\,(h_1(\lambda) + h_2(\lambda))$. It is easy to see that the external blur modifier m_λ satisfies conditions 1) – 3).

Proposition 2. *The external blur modifier* $m_\lambda A = (a_1 - h_1(\lambda), a_1, a_2, a_2 + h_2(\lambda))$ *satisfies condition 4) in the metric* d_δ *if and only if*

$$(h_2(\lambda) - h_2(\tau))\,(1 - 2\delta) \geq (h_1(\lambda) - h_1(\tau))\,(1 + 2\delta) \;\; \textit{for} \;\; m_\lambda A, \; m_\tau A \in L,$$

$$(h_2(\lambda) - h_2(\tau))\,(1 + 2\delta) \leq (h_1(\lambda) - h_1(\tau))\,(1 - 2\delta) \;\; \textit{for} \;\; m_\lambda A, \; m_\tau A \in H$$

$\forall\, 0 \leq \lambda \leq \tau \leq 1$.

Corollary 1. *If the external blur modifier* $m_\lambda A = (a_1 - h_1(\lambda), a_1, a_2, a_2 + h_2(\lambda))$ *satisfies condition 4), then* $|\Delta h_2| \geq |\Delta h_1|$ *for* $m_\lambda A \in L$ *and* $|\Delta h_1| \geq |\Delta h_2|$ *for* $m_\lambda A \in H$.

5 Numerical Example

We will consider an example of ranking by the developed method of articles/ reports of conferences in the conference management system, such as EasyChair (`https://easychair.org`). This system uses a septennial scoring system $x_i \in \{-3, -2, -1, 0, 1, 2, 3\}$, corresponding to the recommendations "strong reject", "reject", "weak reject", "borderline paper", "weak accept", "accept", "strong accept". In addition, the reviewer gives an assessment on a five-fold scale (0.2 – "none", 0.4 – "low", 0.6 – "medium", 0.8 – "high", 1 – "expert") about the degree of confidence in the correctness of his decision: $\lambda_i \in \{0.2, 0.4, 0.6, 0.8, 1\}$.

Data $\left\{(x_i^{(k)}, \lambda_i^{(k)})\right\}_{i=1}^{5}$ of $n = 5$ reviewers on 4 articles are presented in Table 1 (i is the index of the reviewer, k is the index of the article, $k = 1, \ldots, 4$), where $\mathbf{x}^{(k)} = (x_{i_1}^{(k)}, \ldots, x_{i_5}^{(k)})$ is an ascending vector of ratings of all reviewers relative to the article k, $\mathbf{t}^{(k)} = (t_{i_1}^{(k)}, \ldots, t_{i_5}^{(k)})$ is the same vector in a three-gradation scale $\{L, M, H\}$, $L = \{-3, -2, -1\}$, $M = \{0\}$, $H = \{1, 2, 3\}$.

Note that if we take into account only the three-grade recommendations of the reviewers ($L = \{-3, -2, -1\}$, $M = \{0\}$, $H = \{1, 2, 3\}$) and do not take into account the degree of confidence, then we will get the following vectors of estimates $\mathbf{v}(\mathbf{x}^{(k)}) = \left(v_L(\mathbf{x}^{(k)}), v_M(\mathbf{x}^{(k)}), v_H(\mathbf{x}^{(k)})\right)$ for each article $k = 1, \ldots, 4$ (see Table 2).

If, however, estimates close to the medium ± 1 with a low degree of confidence $\lambda \leq 0.6$ are attributed to the class M, then we will obtain the following (extended) vectors of estimates $\mathbf{v}_{\text{ext}}(\mathbf{x}^{(k)})$, $k = 1, \ldots, 4$ and a new ranking (see Table 2).

Table 1. Initial data on the assessments of reviewers.

	paper 1	paper 2	paper 3	paper 4
reviewer 1	$(2, 0.8)$	$(2, 1)$	$(1, 0.8)$	$(0, 0.6)$
reviewer 2	$(1, 1)$	$(2, 0.8)$	$(2, 0.6)$	$(1, 0.4)$
reviewer 3	$(0, 0.8)$	$(0, 0.6)$	$(-1, 0.6)$	$(1, 1)$
reviewer 4	$(3, 0.4)$	$(-1, 0.4)$	$(0, 0.6)$	$(2, 0.2)$
reviewer 5	$(2, 0.6)$	$(1, 0.6)$	$(1, 1)$	$(2, 1)$
$\mathbf{x}^{(k)}$	$(0, 1, 2, 2, 3)$	$(-1, 0, 1, 2, 2)$	$(-1, 0, 1, 1, 2)$	$(0, 1, 1, 2, 2)$
$\mathbf{t}^{(k)}$	(M, H, H, H, H)	(L, M, H, H, H)	(L, M, H, H, H)	(M, H, H, H, H)

Let us now consider fuzzy blurring of point estimates by trapezoidal fuzzy numbers using the technique described above.

Let us put each point estimate $x \in \{-3, -2, -1, 0, 1, 2, 3\}$ in correspondence with the segment: $[0.75(x-1), 0.75x]$ for $x \in \{-3, -2, -1\}$; $[0.75x, 0.75(x+1)]$ for $x \in \{1, 2, 3\}$; $[-0.75, 0.75]$ for $x = 0$. This segment will be the kernel $\ker(\widetilde{x})$ of the trapezoidal fuzzy number $\widetilde{x}$. For example, if $x = 2$ ("weak accept"), then $\ker(\widetilde{x}) = [1.5, 2.25]$.

We will use the modifier of the external blurring $m_\lambda A = (a_1 - h_1(\lambda), a_1, a_2, a_2 + h_2(\lambda))$ of the segment-kernel $A = [a_1, a_2]$, where the blurring functions $h_1(\lambda)$, $h_1(\lambda)$ will be chosen as follows: $h_1(\lambda) = 0$, $h_2(\lambda) = \frac{1-\lambda}{5+\lambda}$ for $A \in L$; $h_1(\lambda) = \frac{1-\lambda}{5+\lambda}$, $h_2(\lambda) = 0$ for $A \in H$; $h_1(\lambda) = h_2(\lambda) = \frac{1-\lambda}{5+\lambda}$ for $A \in M$. Such a modifier satisfies all conditions 1) – 4) of Section 4.

Next, we calculate the vectors of fuzzy estimates $\mathbf{v}(\widetilde{\mathbf{x}}^{(k)}) = \big(v_L(\widetilde{\mathbf{x}}^{(k)}), v_{M^-}(\widetilde{\mathbf{x}}^{(k)}), v_{M^+}(\widetilde{\mathbf{x}}^{(k)}), v_H(\widetilde{\mathbf{x}}^{(k)})\big)$ by formulas (2) ($a = 3$, $b = 1.5$, $\delta = 0.3$) (see Table 2).

Table 2. Results of crisp and fuzzy threshold aggregations.

	paper 1	paper 2	paper 3	paper 4
$\mathbf{v}(\mathbf{x}^{(k)})$	$(0, 1, 4)$	$(1, 1, 3)$	$(1, 1, 3)$	$(0, 1, 4)$
$\mathbf{v}_{\text{ext}}(\mathbf{x}^{(k)})$	$(0, 1, 4)$	$(0, 3, 2)$	$(0, 2, 3)$	$(0, 2, 3)$
$\mathbf{v}(\widetilde{\mathbf{x}}^{(k)})$	$(0, 0.5, 0.5, 2.59)$	$(0, 1.33, 1.33, 1.3)$	$(0, 1.33, 0.5, 1.57)$	$(0, 0.5, 1.33, 1.75)$

The final rankings obtained by the methods of crisp and fuzzy threshold aggregations are given in Table 3.

The above example shows that fuzzy threshold aggregation is more sensitive to ranking than the crisp rule. Some alternatives that were indistinguishable

Table 3. Results of crisp and fuzzy threshold ranking.

	ranking
$\mathbf{v}(\mathbf{x}^{(k)})$	$\varphi(\mathbf{x}^{(1)}) = \varphi(\mathbf{x}^{(4)}) > \varphi(\mathbf{x}^{(3)}) = \varphi(\mathbf{x}^{(2)})$
$\mathbf{v}_{\text{ext}}(\mathbf{x}^{(k)})$	$\varphi(\mathbf{x}^{(1)}) > \varphi(\mathbf{x}^{(4)}) = \varphi(\mathbf{x}^{(3)}) > \varphi(\mathbf{x}^{(2)})$
$\mathbf{v}(\widetilde{\mathbf{x}}^{(k)})$	$\varphi(\mathbf{x}^{(1)}) > \varphi(\mathbf{x}^{(4)}) > \varphi(\mathbf{x}^{(3)}) > \varphi(\mathbf{x}^{(2)})$

with respect to the non-fuzzy threshold aggregation rule began to differ with respect to the fuzzy rule.

6 Conclusion

The paper developed a procedure for threshold ranking of alternatives represented by vectors of fuzzy numbers. The procedure is described for the case of a three-grade fuzzy rating scale but can be generalized to an arbitrary case of m-gradation scales, $m \geq 3$.

In addition, the paper proposes and investigates a procedure for blurring point expert data on information about the degree of confidence of experts in their estimates. Both the general properties of such estimates and the properties in relation to specific blur models are studied.

The specified procedures of fuzzy threshold aggregation and blurring are demonstrated on the example of ranking articles according to the recommendations of reviewers and the degree of their confidence in their recommendations.

In the future, it is of interest to develop the axiomatic of fuzzy threshold aggregation.

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