Conflict Measure of Belief Functions with Blurred Focal Elements on the Real Line*

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Abstract. The paper studies the variation of the conflict measure with blurring of focal elements and discounting of the masses of the belief functions in the framework of the theory of evidence. Blurring of focal elements is modeled using fuzzy sets. Such properties of the conflict measure as the robustness to transformations of the bodies of evidence, the monotonicity and the direction of change are investigated. A numerical example of calculating the measure of conflict, taking into account the blurring of focal elements and discounting of masses for the selection of bodies of evidence for the aggregation of analysts' forecasts regarding the oil price, is considered.

Keywords: Evidence theory · Conflict measure · Blur focal elements.

1 Introduction

Conflict assessment and combining of evidence bodies in the evidence theory is a two-pronged problem. On the one hand, the conflict value must be taken into account when combining evidence. On the other hand, the conflict itself is estimated, as a rule, with the help of aggregation of evidence bodies. The method of conflict estimation and the choice of the combining rule should be consistent with each other in a certain sense [9].

As a rule, the (external) conflict of two bodies of evidence is understood as a quantity proportional to the sum of the products of the masses of non-intersecting (or 'weakly intersecting' with respect to some similarity index) focal elements of this evidence. For example, the conflict in Dempster's rule [2] is the mass of the empty set obtained using the non-normalized conjunctive rule.

Robustness is one of the important requirements for conflict assessment. We will under-stand by robustness the stability of the conflict measure to the 'small' variation of focal elements and their masses. In general, the robustness of calculation the conflict measure can be achieved by applying specialization-generalization procedures [1, 5].

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We will consider bodies of evidence, the focal elements of which are defined on some set $X \subseteq \mathbb{R}$ [11]. The specialization-generalization procedure for such bodies of evidence can be implemented using fuzzy blurring of focal elements. The concept of fuzzy focal elements was considered, for example, in [12, 13].

If the focal elements on the $X \subseteq \mathbb{R}$ are defined by experts, then information on the preferential conservatism or radicalism of expert can also be taken into account using the blur procedure. For example, if one expert predicts the value of shares of a certain company in the interval $A = [40, 45)$, and another in the interval $B = [35, 50)$, then the second evidence can be considered more conservative than the first. If the second expert often gives conservative estimates, then we can assume that the interval B is a support of a fuzzy number-evidence. On the contrary, if the estimates of the first expert are often radical, then we can assume that the interval A is the kernel of a fuzzy number-evidence.

Finally, the blurring procedure of focal elements can be used to account for the reliability of information sources together with a procedure for discounting the masses.

In this paper, we will investigate some properties of the conflict measure of evidence bodies, determined on the $X \subseteq \mathbb{R}$, taking into account blurring of focal elements and discounting of masses. A numerical example of the choice for combining evidence bodies will be considered, taking into account their conflict, reliability and accuracy and using the procedure for blurring of focal elements.

2 Background of the Belief Function Theory

Let X be some set, $\mathcal{A} \subseteq 2^X$ be some finite subset of nonempty sets from X. Some non-negative mass function $m : 2^X \to [0, 1]$, $\sum_{A \in \mathcal{A}} m(A) = 1$ is considered in the theory of evidence [2, 10]. Without loss of generality, we can assume that $m(A) > 0$ for all $A \in \mathcal{A}$. In this case, set $\mathcal{A}$ is called the set of focal elements, and a pair $F = (\mathcal{A}, m)$ is called a body of evidence. Let $\mathcal{F}(X)$ be a set of all bodies of evidence on X. There is one-to-one correspondence between the body of evidence $F = (\mathcal{A}, m)$ and the belief function $Bel(A) = \sum_{B \subseteq A} m(B)$ or the plausibility function $Pl(A) = \sum_{B: A \cap B \neq \emptyset} m(B)$.

The body of evidence (evidence) $F_A = (A, 1)$ (i.e., $\mathcal{A} = \{A\}$, $m(A) = 1$) is called categorical. In particular, the body of evidence F_X is called vacuous.

Then any body of evidence $F = (\mathcal{A}, m)$ can be represented as a convex sum of categorical bodies of evidence: $F = \sum_{A \in \mathcal{A}} m(A) F_A$. The body of evidence is called simple, if $F_A^\zeta = (1 - \zeta) F_A + \zeta F_X$, $\zeta \in [0, 1]$.

In this paper, we will consider evidence bodies on $X \subseteq \mathbb{R}$ [11]. Moreover, we assume that all focal elements of evidence are intervals of the form $[a_1, a_2)$. In this case, the intersection of such sets will also have the form $[a_1, a_2)$. Consequently, we get a new set of focal elements of the same kind when combining the bodies of evidence using conjunctive rules.

Suppose there are two bodies of evidence $F_1 = (\mathcal{A}_1, m_1)$ and $F_2 = (\mathcal{A}_2, m_2)$. It is necessary to assess the conflict between these two bodies of evidence. Traditionally this is done most often with the help of the measure

$$Con_0(F_1, F_2) = \sum_{A \cap B = \emptyset} m_1(A) m_2(B).$$

However, this measure does not take into account the 'weakly intersecting' (i.e. pairs of intersecting focal sets of different bodies of evidence for which the external measure (for example, the length of interval on $\mathbb{R}$) the intersection of the sets is small compared to the external measure of each of these sets) focal elements of the bodies of evidence. The value of the conflict should be a decreasing function of the value of the 'strong' intersection of focal elements with large masses in the general case. Therefore, we will use the measure

$$Con^{\Gamma}(F_1, F_2) = \sum_{A \in \mathcal{A}_1, B \in \mathcal{A}_2} \gamma(A, B) m_1(A) m_2(B), \tag{1}$$

instead of the measure $Con_0(F_1, F_2)$ to account for 'weakly intersecting' focal elements, where $\Gamma = (\gamma(A, B))_{A, B \in \mathcal{A}}$, $\gamma(A, B) = 1 - s(A, B)$ and $s(A, B)$ is a similarity index satisfying the conditions: 1) $0 \le s(A, B) \le 1$; 2) $s(A, B) = 0$, if $A \cap B = \emptyset$; 3) $s(A, A) = 1$ (or weaker condition 3)' $\max_B s(A, B) = s(A, A)$). An example of such an index is the Jaccard index $s(A, B) = |A \cap B| / |A \cup B|$, which we will mainly consider in this article. Note that if $\gamma(A, B) = 1$ in the case $A \cap B = \emptyset$ and $\gamma(A, B) = 0$ in all other cases, then in (1) we get the measure $Con_0(F_1, F_2)$. Some properties of the bilinear conflict measure of the form (1) were investigated in [6].

The conflict measure (1) will be coordinated with the combination of the bodies of evidence $F_1 = (\mathcal{A}_1, m_1)$, $F_2 = (\mathcal{A}_2, m_2)$, according to the rule $F_{1,2} = (\mathcal{A}, m_{1,2}) = F_1 \otimes F_2$, where

$$m_{1,2}(C) = \frac{1}{K} \sum_{A \cap B = C} s(A, B) m_1(A) m_2(B), \tag{2}$$

if $K = 1 - Con^{\Gamma}(F_1, F_2) \neq 0$. This is Zhang's center combination rule [14]. The general structure of the bilinear combination rules was investigated in [7].

The reliability of information sources can be taken into account using Shafer's discounting method [10]: $m^{(\eta)}(A) = \eta m(A)$, if $A \neq X$ and $m^{(\eta)}(X) = 1 - \eta + \eta m(X)$, where $\eta \in [0, 1]$. The change in ignorance after the application of Dempster's rule to the discounted bodies of evidence was studied in [8].

3 Blurring Focal Elements

Let $\widetilde{F} = (\widetilde{\mathcal{A}}, \widetilde{m})$ be a transformation of the body of evidence $F = (\mathcal{A}, m)$, where $\widetilde{\mathcal{A}}$ is a set of fuzzy focal elements, i.e. blurring intervals from the $\mathcal{A}$; $\widetilde{m}$ is a discounted mass function.

We will consider below the important properties of the conflict measure in relation to blur and discounting operations. Let d be some metric on the set of all fuzzy sets (see [4]).

Definition 1. *(robustness). The conflict measure Con will be called robust to small transformations* $\widetilde{\ }$ *of the bodies of evidence, if* $\forall F_1, F_2 \in \mathcal{F}(X)$ *and* $\forall \varepsilon > 0$ $\exists \delta_1, \delta_2 > 0$*:* $\left| Con(\widetilde{F_1}, F_2) - Con(F_1, F_2) \right| < \varepsilon$ $\forall \widetilde{F_1} = (\widetilde{\mathcal{A}}_1, \widetilde{m_1}) \in \mathcal{F}(X)$*:* $d\left(\widetilde{\mathcal{A}}_1, \mathcal{A}_1\right) = \sum_{A \in \mathcal{A}_1} d\left(\widetilde{A}, A\right) < \delta_1$, $d\left(\widetilde{m_1}, m_1\right) = \sum_{A \in \mathcal{A}_1} \left|\widetilde{m}(A) - m(A)\right| < \delta_2$.

Definition 2. *(monotonicity). Let's call the conflict measure Con monotonic (strictly monotonic) with respect to a given transformation* $\widetilde{\ }$ *of the body of evidence, if* $\forall F_1, F_2, F_3 \in \mathcal{F}(X)$*:* $Con(F_1, F_2) \le Con(F_1, F_3) \Rightarrow Con(\widetilde{F_1}, F_2) \le Con(\widetilde{F_1}, F_3)$ *(the corresponding inequalities are strictly).*

If the conflict measure is monotonic, then blurring does not change the order relation with respect to that measure. In particular, in the problem of choosing the least conflicting evidence bodies for combining, the monotonicity of the conflict measure means that these transformations of evidence bodies will not lead to a change in the choice.

It is easy to see that the conflict measure Con_0 is monotonic if the transformation is reduced only to discounting the masses according to Shafer's method.

Definition 3. *(directionality of change). A transformation* $\widetilde{\ }$ *is said to be non-increasing (not decreasing) the conflict measure Con, if* $Con(\widetilde{F_1}, F_2) \le Con(F_1, F_2)$ $(Con(\widetilde{F_1}, F_2) \ge Con(F_1, F_2))$ $\forall F_1, F_2 \in \mathcal{F}(X)$*.*

It is easy to see that discounting the masses by Shafer's method does not increase the degree of conflict Con_0. This observation is interpreted as follows: if we know that the reliability of the information source is low, then this information itself becomes less conflicting with information from other sources.

The properties of monotony and directionality of transformation may not be satisfied for arbitrary conflict measure and transformation of evidence bodies.

Let the number $\eta \in [0, 1]$ characterize the level of reliability of the information source ($\eta = 1$ corresponds to an absolutely reliable source). If the source of information is not entirely reliable ($\eta < 1$), then we will consider blurring of focal elements together with discounting of masses. If $A = [a_1, a_2)$ is a focal element, then $\tilde{A} = A^{(\eta)}$ is a fuzzy number associated with A. We have that $A^{(1)} = A$.

Let the symmetric (L-R)-type fuzzy number [4] $\tilde{A}$ be the blur of the focal element $A = [a_1, a_2)$. It means that the fuzzy number $\tilde{A}$ has a membership function $\mu_{\tilde{A}}$: $\mu_{\tilde{A}}(x) = 1$ for $x \in [x_1, x_2)$, $\mu_{\tilde{A}}(x) = L(x) = \theta\left(\frac{x_1 - x}{\delta|A|}\right)$ for $x \in [x_1 - \delta\left|A\right|, x_1]$, $\mu_{\tilde{A}}(x) = R(x) = \theta\left(\frac{x - x_2}{\delta|A|}\right)$ for $x \in [x_2, x_2 + \delta\left|A\right|]$, where $x_1 \le x_2$, $\delta \in (0, 1)$ and a strictly decreasing integrable function $\theta : [0, 1] \to [0, 1]$ satisfies the conditions $\theta(0) = 1$, $\theta(1) = 0$. The value $\delta = \delta(\eta) > 0$ controls the degree of blur. We assume that $\delta(1) = 0$ and $\delta(\eta)$ are a non-increasing function on $[0, 1]$.

As a result, we get a fuzzy focal element $\tilde{A} = A^{(\eta)}$, which is a blur (more precisely, a δ-blur) of the element A, $\ker \tilde{A} = \{x \in \mathbb{R} : \mu_{\tilde{A}}(x) = 1\} = [x_1, x_2]$ (the core of the fuzzy number $\tilde{A}$), $\operatorname{supp}\tilde{A} = \overline{\{x \in \mathbb{R} : \mu_{\tilde{A}}(x) > 0\}} = [x_1 - \delta\,|A|\,, x_2 + \delta\,|A|]$ (the support of the fuzzy number $\tilde{A}$). Let $\left|\tilde{A}\right| = \int_X \mu_{\tilde{A}}(x)dx$ be the cardinality of a fuzzy set $\tilde{A}$, $\mathrm{EI}[\tilde{A}] = [\mathrm{E[L]}, \mathrm{E[R]}]$, where $\mathrm{E}[L] = \int_{-\infty}^{x_1} x dL(x)$, $\mathrm{E}[R] = -\int_{x_2}^{+\infty} x dR(x)$ (expected interval of the fuzzy number $\tilde{A}$).

If an expert (decision maker, DM) is a source of information, then different blurring strategies, depending on the information about the degree of caution of the DM, are possible:

1) if the DM estimates are too careful (conservative), then $\operatorname{supp}\tilde{A} = A$ (internal blur);

2) if the DM estimates are excessively accurate (radical), then $\ker\tilde{A} = A$ (external blur);

3) if the DM estimates are neutral, then $\mathrm{EI}[\tilde{A}] = \bar{A}$, where EI is the expected interval of a fuzzy number (neutral blur).

The meaning of these conditions is as follows. A cautious expert's assessments are often too imprecise. Therefore, they should be made more accurate ($\operatorname{supp}\tilde{A} = A$) when blurring. On the other hand, the assessments of an careless expert are often overly precise. Therefore, they must be expanded ($\ker\tilde{A} = A$) when blurring.

Lemma 1. *The following properties are valid:*

a) $\ker\tilde{A} = [a_1 + \delta\,|A|\,, a_2 - \delta\,|A|]$, $\delta = \delta(\eta) \in (0, 0.5]$ *for internal δ-blur (i.e.* $\operatorname{supp}\tilde{A} = A$*);*

b) $\operatorname{supp}\tilde{A} = [a_1 - \delta\,|A|\,, a_2 + \delta\,|A|]$, $\delta = \delta(\eta) > 0$ *for external δ-blur (i.e.* $\ker\tilde{A} = A$*);*

c) $\ker\tilde{A} = [a_1 + \delta\,|A|\,\theta_0, a_2 - \delta\,|A|\,\theta_0]$, $\operatorname{supp}\tilde{A} = [a_1 - \delta\,|A|\,(1-\theta_0), a_2 + \delta\,|A|\,(1-\theta_0)]$ *for neutral δ-blur (i.e.* $\mathrm{EI}[\tilde{A}] = \bar{A}$*), where* $\theta_0 = \int_0^1 \theta(s)ds$ *and* $0 < \delta(\eta) \le \frac{1}{2\theta_0}$.

Corollary 1. *If* $d\left(\widetilde{A}, A\right) = \int_X \left|\mu_A(x) - \mu_{\widetilde{A}}(x)\right| dx$*, then:*

a) $d\left(\widetilde{A}, A\right) = 2\delta\,|A|\,(1-\theta_0)$ *for internal δ-blur of the interval A;*

b) $d\left(\widetilde{A}, A\right) = 2\delta\,|A|\,\theta_0$ *for external δ-blur of the interval A;*

c) $d\left(\widetilde{A}, A\right) = 4\delta\,|A|\,\theta_1$ *for neutral δ-blur of the interval A, where* $\theta_1 = \int_{\theta_0}^1 \theta(s)ds$.

Note that $\theta_0 = \frac{1}{2}$, $\theta_1 = \frac{1}{8}$, if $\theta(t) = 1 - t$. The fuzzy number $\tilde{A}$ will be trapezoidal in this case.

Definition 4. *The arrangement of intervals of two sets $\mathcal{A}_1$ and $\mathcal{A}_2$ is called stable if there is such $\delta_0 > 0$ that the nature of the inclusion or intersection of the pairs* $\operatorname{supp} A$-$\operatorname{supp} B$, $\ker A$-$\ker B$ *is preserved for δ-blur $\forall\delta < \delta_0$ and $\forall A \in \mathcal{A}_1$,$B \in \mathcal{A}_2$. Let's call this δ-blur small.*

4 Conflict Variation when Transforming Evidence Bodies

Let ${}^{\eta}F = (\mathcal{A}^{(\eta)}, m^{(\eta)})$,$\eta \in [0,1]$ be the body of evidence obtained as a result of blurring the focal elements and discounting the masses of evidence $F = (\mathcal{A}, m)$. We have that ${}^{1}F = F$.

The conflict measure Con^{Γ} will be robust to small transformations in the evidence bodies due to the continuous dependence of the masses on the discounting parameter η and the Jaccard index $s(A,B)$ on the blurring parameter δ.

Let us first consider the variation of the conflict measure Con^{Γ} when the transformation consists only in discounting the masses. The following statements are true.

Proposition 1. *The condition* $Con^{\Gamma}({}^{\eta}F_1, F_2) \leq Con^{\Gamma}(F_1, F_2)$, $\eta \in [0,1]$ *is satisfied for the body of evidence* F_1 *and* $\forall F_2$ *if and only if the inequality*

$$\sum_{A \in \mathcal{A}_1} s(A,B) m_1(A) \leq s(X,B) \tag{3}$$

is true $\forall B \in \mathcal{A}_2$.

Corollary 2. *The condition* $Con^{\Gamma}({}^{\eta}F_1, F_2) \leq Con^{\Gamma}(F_1, F_2)$, $\eta \in [0,1]$ *is satisfied* $\forall F_1, F_2 \Leftrightarrow s(A_B, B) \leq s(X,B)$ $\forall B \in \mathcal{A}_2$, *where* $A_B = \arg\max_{A \in \mathcal{A}_1} s(A,B)$.

Corollary 3. *If* $s(A,B) \leq s(X,B)$ $\forall A, B$ *or* $s(X,B) = 1$ $\forall B$, *then the inequality* $Con^{\Gamma}({}^{\eta}F_1, F_2) \leq Con^{\Gamma}(F_1, F_2)$, $\forall \eta \in [0,1]$ *is true for arbitrary bodies of evidence* F_1 *and* F_2.

Example 1. Let $s_0(A,B) = \begin{cases} 1, A \cap B \neq \emptyset, \\ 0, A \cap B = \emptyset. \end{cases}$ Then condition (3) will be true, since $s_0(X,B) = 1$ $\forall B$. In this case, we have $Con^{\Gamma}(F_1, F_2) = Con_0(F_1, F_2)$ and we will obtain: $Con_0({}^{\eta}F_1, F_2) \leq Con_0(F_1, F_2)$ $\forall \eta \in [0,1]$.

Example 2. Let $s(A,B) = \frac{|A \cap B|}{|X|}$. Then the condition $s(A,B) \leq s(X,B)$ $\forall A, B$ is satisfied and also the inequality $Con^{\Gamma}({}^{\eta}F_1, F_2) \leq Con^{\Gamma}(F_1, F_2)$, $\forall \eta \in [0,1]$ is true.

The continuous dependence of the conflict measure $Con^{\Gamma}({}^{\eta}F_1, F_2)$ on the discount coefficient η implies that the measure Con^{Γ} will be strictly monotonic for small (close to 1) discounting of the masses.

If focal elements are blurred only, then we have the following proposition.

Proposition 2. *We have in the case of small internal (external) δ-blurring of focal elements:* $0 \leq Con^{\Gamma}({}^{\eta}F_1, F_2) - Con^{\Gamma}(F_1, F_2) \leq \frac{2\delta(\eta)\theta_0}{1 - 2\delta(\eta)\theta_0}$ $\left(\left| Con^{\Gamma}(F_1, F_2) - Con^{\Gamma}({}^{\eta}F_1, F_2) \right| \leq 2\delta(\eta)\theta_0 \left(1 - Con^{\Gamma}(F_1, F_2)\right) \right)$.

5 Conflict Variation when Transforming Categorical Bodies of Evidence

We consider more detailed the conflict variation when transforming categorical bodies of evidence. Let two categorical bodies of evidence F_A and F_B be given. Then $Con^{\Gamma}(F_A, F_B) = 1 - s(A, B)$. We'll consider blurring and discounting the body of evidence F_A. As a result, we get a simple evidence ${}^{\eta}F_A = \eta F_{A^{(\eta)}} + (1 - \eta)F_X$, where $A^{(\eta)}$ is some blur of the focal element A. Then

$$Con^{\Gamma}({}^{\eta}F_A, F_B) = 1 - s(A^{(\eta)}, B)\eta - s(X, B)(1 - \eta).$$

We will evaluate the change in conflict in the case of discounting and blurring of categorical evidence for substantially 'close' focal elements.

Definition 5. *The focal elements A and B are said to be substantially close to each other with respect to the index s, if $s(A, B) \geq \max\{s(A, X), s(X, B)\}$.*

The substantially closeness of the focal elements A and B suggests that they are not only close to each other (the value $s(A, B)$ is large), but also strongly differ from the entire set X. Below, we will consider relations of substantial closeness only with respect to the Jaccard index.

Suppose now that only the focal element A is blurred in the categorical evidence F_A. In this case, we have the following propositions.

Proposition 3. *We have:*

1) $Con^{\Gamma}({}^{\eta}F_A, F_B) \geq Con^{\Gamma}(F_A, F_B)$ in the case of small internal blurring of the element A;

2) in the case of small external blurring of the element A: $Con^{\Gamma}({}^{\eta}F_A, F_B) \geq Con^{\Gamma}(F_A, F_B)$ if $B \subseteq A$ and $Con^{\Gamma}({}^{\eta}F_A, F_B) \leq Con^{\Gamma}(F_A, F_B)$ in all other cases.

Proposition 4. *If the focal element A has the same relative position with the elements B and C (with respect to mutual inclusion or intersection), then the conflict measure Con^{Γ} will be monotonic for small blurring of any nature (internal, external, or neutral) for the triple categorical bodies of evidence F_A, F_B and F_C, i.e. $Con^{\Gamma}({}^{\eta}F_A, F_B) \leq Con^{\Gamma}({}^{\eta}F_A, F_C)$, if $Con^{\Gamma}(F_A, F_B) \leq Con^{\Gamma}(F_A, F_C)$ for any admissible values η.*

Finally, we present some result on the change in the conflict when discounting and blurring categorical bodies of evidence only for the case of a stable location of substantially close focal elements A and B.

Proposition 5. *Let the focal elements A and B be stably located, substantially close and $B \subseteq A$ or $A \subseteq B$. Then the inequality $Con^{\Gamma}({}^{\eta}F_A, F_B) \geq Con^{\Gamma}(F_A, F_B)$ is true for small internal blurs $\Leftrightarrow \delta(\eta) \leq \frac{1-\eta}{2\theta_0} \cdot \frac{s(A,B)-s(X,B)}{s(A,B)-(1-\eta)s(X,B)}$ & $B \subseteq A$ or $\delta(\eta) \leq \frac{1-\eta}{2\theta_0 \eta} \cdot \frac{s(A,B)-s(X,B)}{s(A,B)}$ & $A \subseteq B$.*

6 Numerical Example

Let's consider an example of selection of analysts' forecasts on the cost of Brent crude oil in 2021 for aggregation [3]. Forecast prices provided by 7 major investment banks: BNP Paribas, Citigroup, RBC, JPMorgan, Bank of America, Deutsche Bank, Standard Chartered. Each forecast is an interval $A_i = [a_i, b_i)$, where a_i, b_i are the forecasts of the i-th investment bank for Brent crude oil prices in the 4^{th} quarter of 2020 and in the 4^{th} quarter of 2021, respectively (see Table 1). Thus, each prediction is categorical evidence F_{A_i}, $i = 1, ..., 7$.

We will define the reliability η_i of the categorical body of evidence F_{A_i} as inversely proportional to the deviation of the middle of the predicted interval (the 'mean' value of the categorical evidence F_{A_i}) $E(F_{A_i}) = \frac{a_i+b_i}{2}$ from the current value of the Brent oil price c_0. In this case, the formula $\eta_i = \frac{10+\max d_k - d_i}{15+\max d_k - \min d_k}$ was used, where $d_i = |E(F_{A_i}) - c_0|$. The value c_0 was taken equal to the price of Brent oil as of the date 1.03.2021: $c_0 = 65$. The values of the lengths $l_i = l(A_i) = b_i - a_i$ of the intervals-focal elements characterize the degree of uncertainty in the forecasts. The interval $X = [20, 80]$ was considered as the base set (upper and lower prices for Brent crude oil for the last three years).

Table 1. Boundaries of focal elements, reliability and uncertainty of evidence bodies.

	investment banks	a_i	b_i	η_i	l_i
A_1	BNP Paribas	45	59	0.8	14
A_2	Citigroup	44	56	0.71	12
A_3	RBC	41	55	0.63	14
A_4	JPMorgan	39	52	0.53	13
A_5	Bank of America	47	51	0.67	4
A_6	Deutsche Bank	45	50	0.61	5
A_7	Standard Chartered	35	50	0.41	15

Table 2. The values of the conflict measure Con^{Γ} with discounting and with mixed blur.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
A_1	0.26	**0.5**	0.64	0.75	0.63	0.69	0.76
A_2	0.5	0.35	**0.55**	0.7	**0.56**	0.61	0.73
A_3	0.64	0.55	0.4	0.59	0.59	**0.51**	0.65
A_4	0.75	0.7	0.59	0.44	0.7	0.58	0.57
A_5	0.63	0.56	0.59	0.7	0.4	0.58	0.76
A_6	0.69	0.61	0.51	0.58	0.58	0.42	0.66
A_7	0.76	0.73	0.65	0.57	0.76	0.66	0.43

It seems advisable to use internal blur for 'large' (in length l_i) focal elements and external blur for 'small' focal elements. It can be seen from Table 1 that the focal elements A_5 and A_6 can be considered small and we will use external blur for them. The rest of the focal elements can be considered large and we will use internal blur for them. The values of the conflict measure with discounting and with the described mixed blurring are presented in Table 2. The blur function $\delta(\eta) = 1 - \eta$ was used. In this case, the following pairs of evidence are prioritized for combining according to the principle of minimum conflict: $(F_1, F_2) \succ (F_3, F_6) \succ (F_2, F_3) \sim (F_2, F_5)$.

Let us now consider the aggregation of the highest priority pairs of bodies of evidence (F_1, F_2) and (F_3, F_6) with mixed blur. We will aggregate simple bodies of evidence ${}^{\eta_i}F_{A_i^{(\eta_i)}}$ and ${}^{\eta_j}F_{A_j^{(\eta_j)}}$ using the formula (2): ${}^{\eta_i}F_{A_i^{(\eta_i)}} \otimes {}^{\eta_j}F_{A_j^{(\eta_j)}} = (\mathcal{A}_{i,j}, m_{i,j}) =: F_{i,j}$.

We will evaluate the quality of the combination by finding changes in the characteristics of the aggregated evidence compared to the same characteristics of the aggregated evidence bodies. We will consider such characteristics of evidence as the degree of imprecision, reliability and conflict.

The degree of imprecision of the body of evidence $F = (\mathcal{A}, m)$ will be estimated using the functional $H(F) = \sum_{A \in \mathcal{A}} m(A) |A|$. Let $H_i = H\left({}^{\eta_i}F_{A_i^{(\eta_i)}}\right)$, $H_{i,j} = H(F_{i,j})$.

The reliability of the result of combining the bodies of evidence ${}^{\eta_i}F_{A_i^{(\eta_i)}}$ and ${}^{\eta_j}F_{A_j^{(\eta_j)}}$ (we denote it by $\eta_{i,j}$) will be calculated using the above formula, as a normalized estimate of the distance from the 'average' value of the prediction $E(F_{i,j})$ to the current value of c_0. The 'average' value $E(F)$ of the body of evidence $F = (\mathcal{A}, m)$ is calculated by the formula $E(F) = \sum_{A \in \mathcal{A}} m(A) E(F_A)$ (if A is a fuzzy number, then $E(F_A)$ is equal to the center of gravity of this number: $E(F_A) = \int_X x\mu_A(x)dx / \int_X \mu_A(x)dx$).

In addition, we will find the value of the conflict measure Con^{Γ} between the result of combining $F_{i,j}$ and each of the bodies of evidence ${}^{\eta_i}F_{A_i^{(\eta_i)}}$ and ${}^{\eta_j}F_{A_j^{(\eta_j)}}$. The corresponding values of the conflict measure will be denoted by Con_i^{Γ} and Con_j^{Γ}. Table 3 shows the changes in the characteristics of imprecision, reliability and conflict after aggregation of simple bodies of evidence ${}^{\eta_i}F_{A_i^{(\eta_i)}}$ and ${}^{\eta_j}F_{A_j^{(\eta_j)}}$ for pairs of focal elements A_1,A_2 and A_3,A_6.

Table 3. Changing characteristics after aggregation.

	H_i	H_j	$H_{i,j}$	η_i	η_j	$\eta_{i,j}$	Con^{Γ}	Con_i^{Γ}	Con_j^{Γ}
A_1,A_2	21.11	23.27	13.77	0.8	0.71	0.73	0.5	0.4	0.34
A_3,A_6	27.64	27.51	22.01	0.63	0.61	0.64	0.51	0.65	0.4

It can be seen from this table that the degree of imprecision decreases after combining, reliability increases when combining a pair A_3, A_6 and changes itself in different ways when combining a pair A_1, A_2. The conflict between the result of the combination and the original evidence is reduced. Thus, we get more accurate, equally reliable and less conflicting evidence after combining.

7 Conclusion

The measure of the conflict between the bodies of evidence defined on the real line is considered in this article. The change of this measure in cases of 'blurring' of focal elements and discounting of masses is investigated. These procedures are performed in order to improve the robust properties of the conflict measure, to take into account the caution or optimism of experts as sources of information, and also to take into account the reliability of these sources. Blurring of focal elements is modeled using fuzzy numbers. The properties of the robustness and the monotonicity of the conflict measure, the directionality of change of the conflict measure are considered with regards of small transformation. These properties are being studied for different types of focal element 'blurring', which correspond to varying degrees of expert caution. A numerical example of calculating the values of the conflict measure, taking into account the blurring and discounting, when choosing for the subsequent aggregation of expert forecasts regarding the oil price is considered.

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