Threshold Functions and Operations in the Theory of Evidence

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Research Motivation

- The problem of taking into account the degree of intersection or inclusion of focal elements with a given set is relevant when forming the main functions of the theory of evidence (belief, plausibility, etc.). For example, if a focal element overlaps "weakly" with a given set, then the degree of confidence that this focal element is important in assessing the plausibility of membership of the true alternative in the given set will be small.
- This problem is related to the analysis of the sensitivity of the main functions of the theory of evidence to small changes in focal elements.
- A similar problem of taking into account significant focal elements is relevant when performing operations of aggregating bodies of evidence, assessing conflict, degree of uncertainty, etc.

Outline of Presentation

- Basic Concepts and Notations;
- Threshold Operations of Evidence Theory;
- Threshold Uncertainty and Internal Conflict;
- Threshold Aggregation and External Conflict;
- Summary and Conclusion.

Basic concepts and notations

Let

- $X = \{x_1, ..., x_n\}$ be a finite set;
- 2^X be the set of all subsets on X;
- m: $2^X \to [0,1]$, $\sum_{A \in 2^X} m(A) = 1$, $m(\emptyset) = 0$ is a mass function;
- $\mathcal{A}$ is a set of all focal elements, i.e. $A \in \mathcal{A}$ if $m(A) > 0$;
- $F = (\mathcal{A}, m)$ is the body of evidence (BE);
- $\mathcal{F}(X)$ be the set of all BEs on the X.
- the BE $F = (\mathcal{A}, m)$ can be represented in the form $F = \sum_{A \in \mathcal{A}} m(A) F_A$, where $F_A = (\{A\}, 1)$ is a categorical BE;
- the BE $F = (\mathcal{A}, m)$ uniquely defines the belief function $Bel(A) = \sum\limits_{B \subseteq A} m(B)$ and its dual plausibility function $Pl(A) = 1 - Bel(\neg A) = \sum\limits_{B \cap A \neq \emptyset} m(B)$.

Threshold Functions of Belief and Plausibility

The summation of masses when calculating the plausibility is performed over all focal elements that have a non-empty intersection with a given set.

This sum may include focal elements that have ”small” intersection compared to the measures of the intersecting sets themselves. We will call them insignificant.

We introduce the so-called threshold plausibility function, which takes into account only significant focal elements

$$Pl_h(A) = \sum_{B:s(A,B)>h} m(B), \quad h \in [0,1), \tag{1}$$

where $s(A,B)$ is a measure of similarity, satisfying the conditions:

1) $0 \leq s(A,B) \leq 1$;

2) $s(A,B) = 0 \Leftrightarrow A \cap B = \emptyset$;

3) $s(A,A) = 1 \ \forall A \neq \emptyset$ (or weaker condition $\max_B s(A,B) = s(A,A)$).

Examples of Similarity Measures

a) Jaccard index $J(A,B)=\dfrac{|A\cap B|}{|A\cup B|}$;

b) $s(A,B)=\dfrac{|A\cap B|}{|X|}$;

c) $s(A,B)=\begin{cases} 1, A\cap B\neq\emptyset, \\ 0, A\cap B=\emptyset; \end{cases}$

d) Simpson coefficient $s(A,B)=\dfrac{|A\cap B|}{\min\{|A|,|B|\}}$;

e) Sörensen inclusion measure $s(A,B)=\dfrac{|A\cap B|}{|B|}$;

f) Sörensen coefficient $s(A,B)=\dfrac{2|A\cap B|}{|A|+|B|}$.

Properties of Functions Pl_h

1) $Pl_0 = Pl$;
2) $Pl_{h_1}(A) \leq Pl_{h_2}(A)\ \forall A \in 2^X$ if $h_1 \geq h_2$;
3) $Pl_h(\emptyset) = 0$, but $Pl_h(X) \leq 1$;
4) if the similarity measure $s(A, B)$ is monotone with respect to A (i.e. $A' \subseteq A''$ implies that $s(A', B) \leq s(A'', B)$), then the function Pl_h will be monotonic.

The monotonicity condition for the similarity measure $s(A, B)$ with respect to the first argument is certainly satisfied by measures b), c), e).

Threshold Belief Function Bel_h

We have

$$Bel_h(A) = 1 - Pl_h(\neg A) = \sum_{B: s(\neg A, B) \le h} m(B). \quad (2)$$

In other words, the threshold belief function is formulated taking into account focal sets, which may contain a "small" number of elements not included in the considered set.

Note that the statement $s(\neg A, B) \le h \Rightarrow A \cap B \neq \emptyset$ is not true in the general case. It will be true, for example, for similarity measures c), d), e).
These similarity measures are preferable to use in threshold belief functions.

In general, the agreement condition

$$Bel_h(A) \leq Pl_h(A) \quad \forall A \in \mathcal{A}. \tag{3}$$

may not be fulfilled.
There is a value $h_{max} = \sup\{h : Bel_h(A) \leq Pl_h(A),\ A \in \mathcal{A}\} > 0$ that the condition (3) is true on the interval $[0, h_{max})$ and false on the interval $(h_{max}, 1)$.
We have $Pl_h(X) = 1$ and $Bel_h(\emptyset) = 0$ $\forall h \in [0, h_{max}]$; $Bel_h(X) = 1$ and $Pl_h(\emptyset) = 0$ $\forall h \in [0, 1)$.

Proposition 1.

If $\{B \in \mathcal{A} : s(\neg A, B) \leq h\} \subseteq \{B \in \mathcal{A} : s(A, B) > h\}$ $\forall A \in \mathcal{A}$, then (3) is true.

Corollary.

- $h_{max} = \dfrac{1}{2}$ for indices d) and e);
- $h_{max} = \min\limits_{B\in\mathcal{A}} \dfrac{|B|}{|X| + |B|}$ for index a);
- $h_{max} = \dfrac{\min_{B\in\mathcal{A}} |B|}{2\,|X|}$ for index b);
- $h_{max} = \min\limits_{B\in\mathcal{A}} \dfrac{2\,|B|}{|X| + 2\,|B|}$ for index f).

Proposition 2.

We have for measures a) or e) and for $h \in (0,1)$

$$Bel_h(\{x_i\}) = \sum_{B\in\mathcal{A}: x_i\in B, |B|\le \frac{1}{1-h}} m(B), \quad Pl_h(\{x_i\}) = \sum_{B\in\mathcal{A}: x_i\in B, |B|< \frac{1}{h}} m(B).$$

Example 1

Let $0 = h_1 < \ldots < h_l = 1$ be an ordered set of different values of the similarity measure $s(A, B)$. The values of $Bel_h(A)$ will not change within the intervals $h \in [h_j, h_{j+1})$, $j = 1, \ldots, l-1$.
Then Bel_h can be represented by the matrix $\mathbf{Bel} = (bel_{ij})$, $bel_{ij} = Bel_{h_j}(A_i)$. Let us find the matrices **Bel** and **Pl** for BE

$$F = 0.2F_{\{a,b\}} + 0.3F_{\{a,c\}} + 0.4F_{\{b\}} + 0.1F_{\{a,b,c\}}$$

on $X = \{a, b, c\}$ and the similarity measure e). We have the partition $H = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$ for this measure, $h_{max} = 0.5$. Then

Bel =

	$[0, \frac{1}{3})$	$[\frac{1}{3}, \frac{1}{2})$
$\{a\}$	0	0
$\{b\}$	0.4	0.4
$\{c\}$	0	0
$\{a, b\}$	0.6	0.7
$\{a, c\}$	0.3	0.4
$\{b, c\}$	0.4	0.5

, **Pl** =

	$[0, \frac{1}{3})$	$[\frac{1}{3}, \frac{1}{2})$
$\{a\}$	0.6	0.5
$\{b\}$	0.7	0.6
$\{c\}$	0.4	0.3
$\{a, b\}$	1	1
$\{a, c\}$	0.6	0.6
$\{b, c\}$	1	1

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Threshold Uncertainty

Functional $U_h : \mathcal{F}(X) \to [0,1]$

$$U_h(F) = \frac{1}{2^n - 2} \sum_A (Pl_h(A) - Bel_h(A)), \ ; \ h \in [0, h_{max})$$

has the meaning of the uncertainty value of the BE $F = (\mathcal{A}, m)$ for a given $h \in [0, h_{max})$. The functional $U_h(F)$ does not increase with respect to $h \in [0, h_{max})$.
The value $E_h(F) = U_0(F) - U_h(F)$ will characterize the total error in calculating the uncertainty at a given threshold value $h \in [0, h_{max})$.

Example 2.
The functionals $U_h(F)$ and $E_h(F)$ for the BE from example 1 are equal, respectively

$$U_h(F) = \begin{cases} \frac{13}{30}, & h \in [0, \frac{1}{3}), \\ \frac{1}{3}, & h \in [\frac{1}{3}, \frac{1}{2}). \end{cases} \quad E_h(F) = \begin{cases} 0, & h \in [0, \frac{1}{3}), \\ \frac{1}{10}, & h \in [\frac{1}{3}, \frac{1}{2}). \end{cases}$$

Threshold Internal Conflict

The internal conflict $Con\_in : \mathcal{F}(X) \to [0,1]$ characterizes the degree of unconsolidation of focal elements of the BE. There are different ways to assess internal conflict. For example, this measure [Daniel 2010]:

$$Con\_in(F) = 1 - \max_{1 \le i \le n} Pl(x_i).$$

is popular. But $Con\_in(F) = 0$, if $\bigcap_{A \in \mathcal{A}} A \neq \emptyset$. This "strict" requirement cannot always be justified.

For example, if the electoral college must choose several candidates from the set $\{a, b, c, d, e\}$ and half of the electors indicated candidates $\{a, b, c\}$, and the other half indicated $\{c, d, e\}$, then $Con\_in(F) = 0$. However, such BE must be considered internally conflicting.

If we use the Pl_h instead of the Pl, we obtain

$$Con\_in_h(F) = 1 - \max_{1 \le i \le n} Pl_h(x_i).$$

Optimization Problem

For example, the measure of internal conflict for the above example with electors and for the Jaccard similarity measure would be equal to

$$Con\_in_h(F) = 1 - \max_{1\le i\le n} \sum_{B\in\mathcal{A}: x_i\in B, |B|<\frac{1}{h}} m(B) = \begin{cases} 0, & h\in[0,\frac{1}{3}), \\ 1, & h\in[\frac{1}{3},\frac{1}{2}]. \end{cases}$$

Since $U_h \searrow$, $Con\_in_h \nearrow$ with respect to h, then the optimization problem of finding a threshold $h\in[0,h_{max})$ that would minimize the functional

$$\Phi_h(F) = U_h(F) + \lambda Con\_in_h(F) \to \min, \;\; \lambda > 0$$

can be formulated.

Threshold Aggregation and External Conflict

Suppose that two BEs $F_1 = (\mathcal{A}_1, m_1)$ and $F_2 = (\mathcal{A}_2, m_2)$ are given on the set X. When combining these BEs into a single BE, we will only consider the strongly interacting focal elements. Then the conjunctive threshold aggregation, similar to Dempster's rule, will take the form

$$m_h(A) = \frac{1}{k_h} \sum_{B \cap C = A, s(B,C) > h} m_1(B) m_2(C), \;\; m_h(\emptyset) = 0, \tag{4}$$

where $k_h = \sum_{s(B,C) > h} m_1(B) m_2(C) \neq 0$.

The value

$$Con_h(F_1, F_2) = 1 - k_h = \sum_{s(B,C) \leq h} m_1(B) m_2(C) \tag{5}$$

has the meaning of a threshold measure of external conflict between BEs.

Generalization

The following more general aggregation rule and conflict measure can be considered instead of (4) and (5)

$$\tilde{m}_h(A) = \frac{1}{\tilde{k}_h} \sum_{B \cap C = A, s(B,C) > h} s(B,C) m_1(B) m_2(C), \quad \tilde{m}_h(\emptyset) = 0, \qquad (6)$$

where $\tilde{k}_h = \sum_{s(B,C)>h} s(B,C) m_1(B) m_2(C) \neq 0$

$$\widetilde{Con}_h(F_1, F_2) = 1 - \tilde{k}_h =$$

$$= \sum_{s(B,C) \leq h} m_1(B) m_2(C) + \sum_{s(B,C) > h} (1 - s(B,C)) \, m_1(B) m_2(C). \qquad (7)$$

The first term in (7) takes into account the masses of non-intersecting or weakly intersecting focal elements.
The masses of strongly intersecting focal elements in the second term are taken into account with low weight.

Example 2

Let three BEs be given on $X = \{a, b, c\}$:
$F_1 = 0.2F_{\{a,b\}} + 0.3F_{\{a\}} + 0.4F_{\{b\}} + 0.1F_{\{a,b,c\}}$,
$F_2 = 0.6F_{\{b,c\}} + 0.4F_{\{a,b,c\}}$,
$F_3 = 0.4F_{\{a\}} + 0.4F_{\{b\}} + 0.2F_{\{a,b,c\}}$.
Suppose we select two BEs with the least external conflict for subsequent aggregation. We have the following values of the conflict measure Con_h for each pair and for the Jaccard similarity:

h	(F_1, F_2)	(F_1, F_3)	(F_2, F_3)
$\left[0, \frac{1}{3}\right)$	0.18	0.28	0.24
$\left[\frac{1}{3}, \frac{1}{2}\right)$	0.58	0.44	0.56
$\left[\frac{1}{2}, \frac{2}{3}\right)$	0.82	0.72	0.8
$\left[\frac{2}{3}, 1\right)$	0.96	0.82	0.92

If we use the usual ($h = 0$) measure of external conflict, then the pair F_1, F_2 will have the least conflict. But if we want to take into account (weakly) overlapping focal elements when assessing conflict, then we can use the integral characteristic of conflict. For example,

$$ICon_w(F_1, F_2) = \int_0^1 w(h) Con_h(F_1, F_2)\, dh,$$

where the weight $w(h) \geq 0$ satisfies the condition $\int_0^1 w(h)\, dh = 1$.

We will get for BEs from example and $w(h) = 1$:
$ICon_w(F_1, F_2) \approx 0.613$, $ICon_w(F_1, F_3) = 0.56$, $ICon_w(F_2, F_3) \approx 0.613$.
The pair F_1, F_3 will be preferable.

If we use the weight $w(h) = \frac{3}{2} - h$, we obtain $ICon_w(F_1, F_2) \approx 0.523$, $ICon_w(F_1, F_3) \approx 0.607$, $ICon_w(F_2, F_3) \approx 0.534$. The pair F_1, F_2 will be preferable.

Summary and Conclusion

- the main advantage of describing BEs using threshold functions is that we can control the degree of uncertainty and conflict in such a description;
- the set of all represents with different thresholds gives us a more complete description of BEs and their aggregation;
- the problems of finding the optimal threshold at which a compromise is achieved between the accuracy of the description and uncertainty, between uncertainty and internal conflict, etc. can be posed.

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Thanks for you attention

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