Evidence-Based Aggregation and Ranking in an Ordinal Scale

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Research Motivation

- We will consider the problem of aggregating information and ranking alternatives in the case of an ordinal scale. There are several aggregation methods in such a scale [Aleskerov 1999].
- It is desirable to consider the degree of inconsistency of information coming from different sources, the degree of its uncertainty, the reliability of these sources, when developing aggregation rules. All these features are well modeled in the Dempster – Shafer theory of evidence [Dempster 1967, Shafer 1976].
- But alternatives are considered on an unordered set in the classical theory of evidence. Recently, evidential structures have also been considered on ordered base sets, on preference structures [Zhang et al 2018, Zhang et al 2021, Zhang & Deng 2021, Martin 2022].
- We will consider the problem of ranking expert assessments in the framework of the theory of evidence.

Outline of Presentation

- Necessary Information from the Theory of Evidence;
- General Scheme of Evidence-Based Aggregation and Ranking in an Ordinal Scale;
  - Formation of BE of Expert Assessments;
  - Entropy of a BE on an Ordered Set;
  - Aggregation of a BE of Expert Assessments;
  - Ranking of Aggregate Estimates;
- Numerical Example;
- Summary and Conclusion.

Information from the Theory of Evidence

Let [Shafer 1976]:

- $T = \{t_1, ..., t_s\}$ be some finite set;
- 2^T is the power set of T;
- $m : 2^T \to [0,1]$, $\sum_{A \in 2^T} m(A) = 1$, $m(\emptyset) = 0$ is the basic belief assignment (mass function);
- $\mathcal{A} = \{A\}$ is the set of all focal elements, i. e. $A \in \mathcal{A}$ if $m(A) > 0$;
- $F = (\mathcal{A}, m)$ is the body of evidence (BE);
- $\mathcal{F}(T)$ is the set of all BE on T;
- $F_A = (\{A\}, 1)$ is called **categorical** BE. In particular, $F_T = (\{T\}, 1)$ is called **vacuous** BE;
- an arbitrary BE $F = (\mathcal{A}, m)$ can be represented as $F = \sum_{A \in \mathcal{A}} m(A) F_A$;
- $F_A^\alpha = \alpha F_A + (1 - \alpha) F_T$, $\alpha \in (0,1)$ is called **simple** BE;
- the BE is said to be **non-dogmatic** if $m(T) > 0$.

We will consider BE on an ordered set $T = \{t_1, ..., t_s\}$, $t_1 < \ldots < t_s$. We will assume that only ordered sequential sets of elements from T (i.e., sets of the form $A = \{t_i, t_{i+1}, \ldots, t_{i+r}\} \subseteq T$) can be focal elements. The set of all such subsets will be denoted by $\left(2^T\right)_{od} \subseteq 2^T$.

The BE $F = (\mathcal{A}, m)$ can be transformed into the **pignistic probability** Bet_F:

$$Bet_F(t_i) = \sum_{A \in \mathcal{A},\ t_i \in A} \frac{m(A)}{|A|}, \quad i = 1, \ldots, s.$$

The amount of ignorance of the information contained in the BE $F = (\mathcal{A}, m)$ will be estimated using Deng's entropy [Deng 2016]

$$DE(F) = -\sum_{A \in \mathcal{A}} m(A) \log_2 \left(\frac{m(A)}{2^{|A|} - 1} \right),$$

which is approximately equal to the sum of the Shannon entropy $-\sum_A m(A) \log_2 m(A)$ and the measure of imprecision $\sum_A m(A) \left|A\right|$.

General Scheme of Evidence-Based Ranking

Let X be a set of alternatives, which are represented by vectors $\mathbf{x} = (x_1, \ldots, x_n)$, where x_i are scores on an ordinal scale $T = \{t_1, \ldots, t_s\}$, $t_1 < \ldots < t_s$, which correspond to the terms: t_1 – "very low", t_2 – "low", etc.

Let us assume that the elements h_i are also known, given in an ordinal or numerical scale, characterizing the degree of confidence of the DM in the correctness of his decision regarding the estimate $x_i \in T$.

It is necessary to aggregate information from experts and rank alternatives. The proposed method consists of the following steps:

1. formation of BE of individual expert assessments, taking into account information about their confidence;
2. aggregation of generated evidence corpora for each alternative using combination rules from evidence theory;
3. ranking of aggregate estimates.

Formation of BE of Expert Assessments

We will assign a simple BE to each assessment $x \in T$ and the degree of confidence h of the DM in the correctness of his decision:

$$F_x = \mu(h) F_{A(x,h)} + (1 - \mu(h)) F_T, \tag{1}$$

where $A(x, h)$ is the blur function of the value $x \in T$ depending on the confidence h of the estimate, $\mu(h)$ is the mass discounting function. We will assume that $\mu(h) = h$ in the example below.

Information in the form of a coefficient $h \in [0, 1]$ about the degree of confidence of the DM in the correctness of his decision can be used to blur the point estimate $x \in T$.

Let's assume that $h=0$ corresponds to absolute uncertainty, $h=1$ corresponds to absolute confidence in one's decision. The lower the degree of confidence, the greater should be the degree of blur.

In general, the blur function $A(x,h): T\times[0,1]\to\left(2^T\right)_{od}$ must satisfy the following conditions:

1. $A(x,1)=\{x\}\ \forall x\in T$;
2. $A(x,0)=T\ \forall x\in T$;
3. $A(x,h_2)\subseteq A(x,h_1)\ \forall x\in T$ if $h_1\le h_2$;
4. $A(t_j,h)=\{t_{j-p},\ldots,t_j,\ldots,t_{j+r}\}$, where $p\le r$ if $t_j<t_k$ and $p\ge r$ if $t_j>t_k$.

Condition 4) means that the set $A(x,h)$ have a greater bias towards the neutral estimate than towards the "extreme" elements when the element $x\in T$ is blurred, if there is a neutral estimate $t_k\in T$ in the rank scale T.

Entropy of a BE on an Ordered Set

If there is a neutral estimate $t_k \in T$ in the rank scale T, then two equally powerful focal sets A and B with the same masses will make different contributions to the entropy depending on the location of their elements relative to the neutral estimate. If the elements are symmetrically located relative to the neutral estimate in the A, but not in the B, then the contribution of the A to the final entropy must be greater than the contribution of the B.

Let us introduce the index $I(A) \in [I_{\min}, I_{\max}]$, which characterizes the asymmetry of the arrangement of elements with respect to the neutral estimate. Let $A_- = \{t \in A : t < t_k\}$, $A_+ = \{t \in A : t > t_k\}$. The index I must satisfy the following conditions:

1. $I(A) = I_{\min}$ if $A_- = \emptyset \vee A_+ = \emptyset$;
2. $I(A') \leq I(A'')$ if $|A'| = |A''|$ and $||A'_+| - |A'_-|| \geq ||A''_+| - |A''_-||$.

For example, the index

$$I(A) = \frac{2\min\{|A_-|, |A_+|\}}{||A_-| - |A_+|| + 1} \in [0, |A| - 1]$$

satisfies these conditions. It is easy to see that:

a) $I(A) = I_{\min} = 0 \Leftrightarrow A_- = \emptyset \vee A_+ = \emptyset$ (in particular $I(A) = 0$ if the A is a singleton);

b) if $A \neq \emptyset$ and $|A_-| = |A_+|$, then $I(A) = I_{\max} = |A| - 1$.

This index will be used to modify the Deng entropy

$$DEM(F) = -\sum_{A \in \mathcal{A}} m(A) \log_2 \left(\frac{m(A)}{2^{|A|+I(A)} - 1} \right).$$

In this case, the measure of average asymmetry $\sum_{A \in \mathcal{A}} m(A) I(A)$ is approximately added to the Deng entropy. It is easy to see that $DEM(F) \geq DE(F)$.

Aggregation of a BEs of Expert Assessments

Further, all BEs F_{x_i} corresponding to the vector of assessments $\mathbf{x} = (x_1, \ldots, x_n)$ are aggregated in accordance with some combination rule $\otimes_R$: $\underbrace{\mathcal{F}(T) \times \ldots \times \mathcal{F}(T)}_{n} \to \mathcal{F}(T)$. As a result, we get the BE

$$F_{\mathbf{x}} = \bigotimes_{i=1}^{n}{}_R\, F_{x_i}.$$

We will consider conjunctive rules (the Dempster rule, the Yager rule, etc.). The non-normalized conjunctive rule for combining BEs $F_i = (\mathcal{A}_i, m_i)$, $i = 1, \ldots, n$ has the form $F_\cap = \bigotimes_{i=1}^{n}{}_\cap\, F_i = \left(\mathcal{A} \cup \{\emptyset\}, m_{\bigcap}\right)$, where

$$m_\cap(A) = \sum_{B_1 \cap \ldots \cap B_n = A} m_1(B_1) \ldots m_n(B_n).$$

The value

$$Con(F_1,\dots,F_n) = m_\cap(\emptyset) = \sum_{B_1\cap\ldots\cap B_n=\emptyset} m_1(B_1)\dots m_n(B_n) \in [0,1]$$

is a measure of the inconsistency of information provided by BEs. To obtain a BE $F = (\mathcal{A}, m)$, it is necessary to redistribute the mass $m_\cap(\emptyset)$ over other focal elements.

The uniform redistribution of mass $m_\cap(\emptyset)$ over all focal elements is carried out in the classical **Dempster rule** $\otimes_D$:

$$m_D(A) = \frac{1}{1 - m_\cap(\emptyset)} m_\cap(A) \;\text{ if }\; A \neq \emptyset \quad\text{and}\quad m_D(\emptyset) = 0.$$

If $m_\cap(\emptyset) = 1$ (absolute conflict), then Dempster's rule does not apply.

The entire value of $m_\cap(\emptyset)$ is added to $m_\cap(T)$, increasing the weight of "not knowing" in **Yager's rule** $\otimes_Y$:

$$m_Y(A) = m_\cap(A) \;\text{ if }\; A \neq \emptyset, T, \;\; m_Y(\emptyset) = 0, \;\; m_Y(T) = m_\cap(T) + m_\cap(\emptyset).$$

Ranking of Aggregate Estimates

Let $\{F_{\mathbf{x}}\}_{\mathbf{x}\in X}$ be a set of BEs; $Bet_{F_{\mathbf{x}}}$ is the pignistic probability corresponding to the BE $F_{\mathbf{x}}$.

We will use the concept of an interval median $\left[\underline{me_{\mathbf{x}}}, \overline{me_{\mathbf{x}}}\right]$ to rank estimates $\{\mathbf{x}\}$ using the resulting BEs. The boundaries $\underline{me_{\mathbf{x}}}$, $\overline{me_{\mathbf{x}}}$ of this interval are defined as follows. Let us consider two sets

$$\underline{I_{\mathbf{x}}}=\left\{t_k : \sum_{i=1}^{k} Bet_{F_{\mathbf{x}}}(t_i) \le \frac{1}{2}, Bet_{F_{\mathbf{x}}}(t_k) \neq 0\right\}, \overline{I_{\mathbf{x}}}=\left\{t_k : \sum_{i=k}^{s} Bet_{F_{\mathbf{x}}}(t_i) \le \frac{1}{2}, Bet_{F_{\mathbf{x}}}(t_k) \neq 0\right\}.$$

Let $\Delta_{\mathbf{x}} = X \setminus \left(\underline{I_{\mathbf{x}}} \cup \overline{I_{\mathbf{x}}}\right)$. Three cases are possible:

1) if $|\Delta_{\mathbf{x}}| = 0$, then we set $\underline{me_{\mathbf{x}}} = \sup \underline{I_{\mathbf{x}}}$, $\overline{me_{\mathbf{x}}} = \inf \overline{I_{\mathbf{x}}}$;
2) if $|\Delta_{\mathbf{x}}| = 1$, then we set $\underline{me_{\mathbf{x}}} = \overline{me_{\mathbf{x}}} = t \in \Delta_{\mathbf{x}}$;
3) if $|\Delta_{\mathbf{x}}| > 1$, then either $Bet_{F_{\mathbf{x}}}(t) = 0\ \forall t \in \Delta_{\mathbf{x}}$ and we set $\underline{me_{\mathbf{x}}} = \sup \underline{I_{\mathbf{x}}}$, $\overline{me_{\mathbf{x}}} = \inf \overline{I_{\mathbf{x}}}$ or there is a single element $t_0 \in \Delta_{\mathbf{x}}$: $Bet_{F_{\mathbf{x}}}(t_0) > 0$ and let $\underline{me_{\mathbf{x}}} = \overline{me_{\mathbf{x}}} = t_0 \in \Delta_{\mathbf{x}}$.

Further, we will consider the set of vectors of "point" medians

$$\mathrm{M}_X = \left\{ (me_{\mathbf{x}})_{\mathbf{x}\in X} : \underline{me_{\mathbf{x}}} \le me_{\mathbf{x}} \le \overline{me_{\mathbf{x}}}, \mathbf{x} \in X \right\}.$$

Then we rank the alternatives X according to the ranking of these medians for each fixed set of medians $\mathbf{me} = (me_{\mathbf{x}})_{\mathbf{x}\in X} \in \mathrm{M}_X$: $\mathbf{a} \succ_{\mathbf{me}} \mathbf{b}$ if $me_{\mathbf{a}} > me_{\mathbf{b}}$ and $\mathbf{a} \sim_{\mathbf{me}} \mathbf{b}$ if $me_{\mathbf{a}} = me_{\mathbf{b}}$. As a result, we get a set of particular rankings (preferences) on X.

The final ranking of alternatives is carried out by applying some preference aggregation rule. A wide choice of such rules is provided by social choice theory. For example, Borda's rule can be used.

Numerical Example

Let us illustrate the proposed scheme for aggregating expert assessments and ranking on the data of reviewing articles in the EasyChair conference management system (`https://easychair.org`).

This system uses a seven-rank ordinal rating scale $T = \{t_1, \ldots, t_7\}$. These ranks t_i, $i = 1, \ldots, 7$ meet the recommendations (in ascending order) "strong reject", "reject", "weak reject", "borderline paper", "weak accept", "accept", "strong accept". In addition, the reviewer gives an assessment on a five-rank scale (h_1 – "none", h_2 – "low", h_3 – "medium", h_4 – "high", h_5 – "expert") about the degree of confidence in the correctness of his decision: $h_1 < \ldots < h_5$. For simplicity, we will assume that the degrees of confidence are given on a numerical scale by the formula $h_j = 0.2j$, $j = 1, \ldots, 5$.

Point data $\left\{\left(x_r^{(k)}, h_r^{(k)}\right)\right\}_{r=1}^{3}$ of $n = 3$ reviewers regarding 4 papers $\{a_1, \ldots, a_4\}$ are presented in Table (r is the index of the reviewer, k is the index of the article), where $x_r^{(k)} \in T$.

	paper a_1	paper a_2	paper a_3	paper a_4
reviewer 1	$(t_6, 0.8)$	$(t_6, 1)$	$(t_5, 0.8)$	$(t_4, 0.6)$
reviewer 2	$(t_5, 1)$	$(t_5, 0.4)$	$(t_6, 0.6)$	$(t_5, 0.4)$
reviewer 3	$(t_4, 0.8)$	$(t_5, 0.6)$	$(t_3, 0.6)$	$(t_5, 1)$

Let us apply the procedure of blurring point estimates $x_r^{(k)}$ taking into account the degrees of confidence $h_r^{(k)}$. As a result, we get simple BE (see Table of focal elements and their masses $\left(A\left(x_r^{(k)}, h_r^{(k)}\right), h_r^{(k)}\right)$). The blur functions themselves are not given due to their cumbersomeness.

	paper a_1	paper a_2	paper a_3	paper a_4
reviewer 1	$(\{t_5, t_6\}, 0.8)$	$(\{t_6\}, 0.95)$	$(\{t_4, t_5\}, 0.8)$	$(\{t_3, t_4, t_5\}, 0.6)$
reviewer 2	$(\{t_5\}, 0.95)$	$(\{t_3, t_4, t_5, t_6\}, 0.4)$	$(\{t_5, t_6, t_7\}, 0.6)$	$(\{t_3, t_4, t_5, t_6\}, 0.4)$
reviewer 3	$(\{t_4\}, 0.8)$	$(\{t_4, t_5, t_6\}, 0.6)$	$(\{t_2, t_3, t_4\}, 0.6)$	$(\{t_5\}, 0.95)$
Con	0.79	0	0.36	0

The values of the measure of conflict $Con(F_{1k}, F_{2k}, F_{3k})$ of assessments of all reviewers for each article are given in the last line.

Let us apply the Dempster rule to aggregate the BEs of all reviewers for each article. We get the following results $F_k = \bigotimes_{r=1}^{3}{}_D F_{rk}$:

$$F_1 = 0.038F_{\{t_4\}} + 0.914F_{\{t_5\}} + 0.038F_{\{t_5,t_6\}} + 0.01F_T,$$

$$F_2 = 0.95F_{\{t_6\}} + 0.03F_{\{t_4,t_5,t_6\}} + 0.008F_{\{t_3,t_4,t_5,t_6\}} + 0.012F_T,$$

$$F_3 = 0.3F_{\{t_4\}} + 0.3F_{\{t_5\}} + 0.2F_{\{t_4,t_5\}} + 0.075F_{\{t_2,t_3,t_4\}} + 0.075F_{\{t_5,t_6,t_7\}} + 0.05F_T,$$

$$F_4 = 0.95F_{\{t_5\}} + 0.03F_{\{t_3,t_4,t_5\}} + 0.008F_{\{t_3,t_4,t_5,t_6\}} + 0.012F_T.$$

Pignistic probabilities $Bet_{F_k}(t_i)$ for BEs aggregated according to Dempster's rule are given in Table

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	DEM
F_1	0	0	0.004	0.04	0.934	0.022	0	0.73
F_2	0	0	0.007	0.014	0.014	0.965	0	0.63
F_3	0.008	0.032	0.032	0.432	0.432	0.032	0.032	3.67
F_4	0	0	0.016	0.014	0.965	0.005	0	0.7

The values of the modified Deng entropy are given in the last column.

All medians of these distributions will be point and equal, respectively: $me_1 = \{t_5\}$, $me_2 = \{t_6\}$, $me_3 = \{t_4\}$, $me_4 = \{t_5\}$. Therefore, the ranking of articles will be as follows:

$$a_2 \succ_{\mathbf{me}} a_1 \sim_{\mathbf{me}} a_4 \succ_{\mathbf{me}} a_3.$$

Let's compare this result with the linear convolution of the criteria. We will assume that the ratings of the reviewers are given on a numerical scale $t_i = i$, $i = 1, \ldots, 7$. Then the linear convolution of criteria for each article with weights $h_r^{(k)}$, $r = 1, 2, 3$ will take the form

$$C(a_k) = \sum_{r=1}^{3} h_r^{(k)} x_r^{(k)}.$$

We will get the following results for our data: $C(a_1) = 12.75$, $C(a_2) = 10.7$, $C(a_3) = 9.4$, $C(a_4) = 9.15$. Thus, the ranking of articles in this case will be as follows:

$$a_1 \succ a_2 \succ a_3 \succ a_4.$$

The difference in rankings can be explained by the fact that the reviews a_1 and a_3 articles are highly conflicting. And this is taken into account when aggregating with conjunctive rules. In addition, low weight estimates significantly reduce the value of linear convolution. While a small weight only "blurs" the reviewer's assessment during evidence-based aggregation.

Summary and Conclusion

- an evidence-based procedure for aggregating and ranking expert information given in an ordinal scale has been proposed;
- the aggregation of BE of expert assessments is based on the use of conjunctive rules of combining;
- the ranking is based on the calculation of (possibly interval) medians corresponding to the pignistic probabilities of the BEs;
- the method demonstrated a certain stability of the result to the choice of the type of conjunctive aggregation rule;
- the proposed method does not underestimate expert estimates with low confidence, but only blurs them more, unlike the method of linear convolution of criteria;
- the use of evidence theory aggregation rules makes it possible to take into account the conflicting nature of expert assessments.

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Thanks for you attention

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