Fuzzy Threshold Aggregation

Alexander Lepskiy

National Research University – Higher School of Economics,
Moscow, Russia

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Research Motivation

- Let us consider the classical problem of aggregation of individual preferences. There are many rules for such aggregation.
- In some cases, aggregation rules should be non-compensatory. This implies that low scores on one criterion cannot be compensated for by high scores on others.
- The so-called threshold rule [Aleskerov et al 2010, Aleskerov & Yakuba 2007] is one of the popular aggregation rules that has a non-compensatory property.
- In some cases, the characteristics of alternatives may be fuzzy. Then the problem of generalizing the threshold aggregation rule to the case of fuzzy data is relevant.

Outline of Presentation

- Non-Fuzzy Formulation of the Threshold Aggregation Problem;
- The Problem of Threshold Aggregation with Fuzzy Data;
- General Scheme of Threshold Fuzzy Ranking;
- Formation of Fuzzy Estimates from Point Data;
- Numerical Example;
- Summary and Conclusion.

Non-Fuzzy Formulation of the Threshold Aggregation Problem

The problem of ranking alternatives of a set X of evaluated by n criteria in a **three-gradation scale** is being considered. The alternatives are represented by vectors: $\mathbf{x} = (x_1, \ldots, x_n)$, where $x_i \in \{1, 2, 3\}$. It is required to find an operator $\varphi_n = \varphi : X \to \mathbb{R}$ that satisfies the conditions [Aleskerov & Yakuba 2007]:

1) **Pareto-domination**:
   if $\mathbf{x}, \mathbf{y} \in X$ and $x_i \geq y_i \quad \forall i,\ \exists s:\ x_s > y_s$, then $\varphi(\mathbf{x}) > \varphi(\mathbf{y})$;
2) **pairwise compensability of criteria**:
   if $\mathbf{x}, \mathbf{y} \in X$ and $v_k(\mathbf{x}) = v_k(\mathbf{y})\ k = 1, 2$, then $\varphi(\mathbf{x}) = \varphi(\mathbf{y})$, where $v_k(\mathbf{x}) = |\{i : x_i = k\}|$ is the number of estimates of k in the alternative $\mathbf{x}$, $k = 1, 2, 3$;
3) **threshold noncompensability**:
   $\varphi(\underbrace{2, \ldots, 2}_{n}) > \varphi(\mathbf{x})\ \forall \mathbf{x} \in X\colon \exists s:\ x_s = 1$;

4) **the reduction axiom**:
if $\forall \mathbf{x}, \mathbf{y} \in X\ \exists s:\ x_s = y_s$, then
$\varphi_n(\mathbf{x}) > \varphi_n(\mathbf{y}) \Leftrightarrow \varphi_{n-1}(\mathbf{x}_{-s}) > \varphi_{n-1}(\mathbf{y}_{-s})$,
where $\mathbf{x}_{-s} = (x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_n)$.

It is shown that the **lexicographic aggregation rule** is a solution to this problem:

$$\varphi(\mathbf{x}) > \varphi(\mathbf{y}) \Leftrightarrow$$

$$\exists j \in \{1, 2\} : v_k(\mathbf{x}) = v_k(\mathbf{y}) \quad \forall k \leq j \ \text{ and } \ v_{k+1}(\mathbf{x}) < v_{k+1}(\mathbf{y}).$$

This problem was generalized in [Aleskerov et al 2010] to the case of m-gradation scales, $m \geq 3$.

Threshold Aggregation with Fuzzy Data

Let us now assume that the alternatives are represented by vectors of fuzzy numbers $\widetilde{\mathbf{x}} = (\widetilde{x_1}, \ldots, \widetilde{x_n})$.

Each fuzzy number belongs to one of three classes: the **low score class** L, the **median score class** M or the **high score class** H.

We will assume that the supports of fuzzy estimators are located on the segments:

- $[-a, 0]$, $a > 0$ for estimators of the class L;
- $[0, a]$, $a > 0$ for estimators of the class H;
- $[-b, b]$, $0 < b < a$ for estimators of the class M.

Crisp numbers $L_0 = -a$, $M_0 = 0$ and $H_0 = a$ are reference elements.

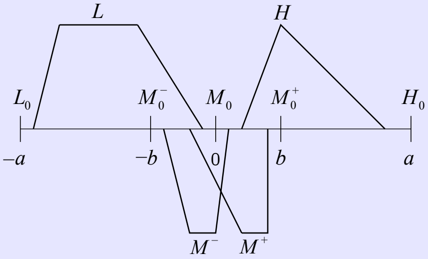
We will consider a set of median positive estimates

$$M^+ = \{\widetilde{x} \in M : d(\widetilde{x}, M_0^+) \leq d(\widetilde{x}, M_0^-)\}$$

and a set of median negative estimates

$$M^- = \{\widetilde{x} \in M : d(\widetilde{x}, M_0^-) \leq d(\widetilde{x}, M_0^+)\},$$

where $M_0^- = -b$, $M_0^+ = b$ are the reference estimates of subclasses M^- and M^+, respectively.



General Scheme of Threshold Fuzzy Ranking

The following steps are performed for each alternative $\widetilde{\mathbf{x}} = (\widetilde{x_1}, \ldots, \widetilde{x_n})$ and each class $S \in \{L, M^-, M^+, H\}$.

1. The distances $d(\widetilde{x}_i, S_0)$ are calculated $\forall \widetilde{x}_i \in S$ (the values $d(\widetilde{x}_i, M_0)$, $d\left(\widetilde{x}_i, M_0^{\pm}\right)$ are calculated for the class M), $i = 1, \ldots, n$.
2. The value
$$F_S(\widetilde{x}_i) = \psi(d(\widetilde{x}_i, S_0))$$
characterizes the normalized degree of confidence that the estimate $\widetilde{x}_i \in S$, where nonincreasing function $\psi : [0, +\infty) \to [0, 1]$ satisfies the condition $\psi(0) = 1$.
3. The value
$$v_S(\widetilde{\mathbf{x}}) = \sum_{\widetilde{x}_i \in S} F_S(\widetilde{x}_i), \quad S \in \{L, M^-, M^+, H\}$$
characterizes the cardinality of the set of fuzzy estimates of the class S.

4. Let's apply the lexicographic aggregation rule:

$$\varphi\left(\widetilde{\mathbf{x}}\right) > \varphi\left(\widetilde{\mathbf{y}}\right) \Leftrightarrow$$

$$v_L\left(\widetilde{\mathbf{x}}\right) < v_L\left(\widetilde{\mathbf{y}}\right)$$

or

$$v_L\left(\widetilde{\mathbf{x}}\right) = v_L\left(\widetilde{\mathbf{y}}\right), \;\; v_{M^-}\left(\widetilde{\mathbf{x}}\right) < v_{M^-}\left(\widetilde{\mathbf{y}}\right)$$

or

$$v_L\left(\widetilde{\mathbf{x}}\right) = v_L\left(\widetilde{\mathbf{y}}\right), \;\; v_{M^-}\left(\widetilde{\mathbf{x}}\right) = v_{M^-}\left(\widetilde{\mathbf{y}}\right), \;\; v_{M^+}\left(\widetilde{\mathbf{x}}\right) < v_{M^+}\left(\widetilde{\mathbf{y}}\right)$$

or

$$v_L\left(\widetilde{\mathbf{x}}\right) = v_L\left(\widetilde{\mathbf{y}}\right), \;\; v_{M^-}\left(\widetilde{\mathbf{x}}\right) = v_{M^-}\left(\widetilde{\mathbf{y}}\right),$$
$$v_{M^+}\left(\widetilde{\mathbf{x}}\right) = v_{M^+}\left(\widetilde{\mathbf{y}}\right), \;\; v_H\left(\widetilde{\mathbf{x}}\right) < v_H\left(\widetilde{\mathbf{y}}\right).$$

We will consider the value

$$d_\delta(A,B) = |Val(A) - Val(B)| + \delta\,|Am(A) - Am(B)|\,, \;\; 0 < \delta \le \tfrac{1}{2}$$

as the distance between fuzzy numbers A and B, where

$$Val(A) = \tfrac{1}{2}\int_0^1 (l_A(\alpha) + r_A(\alpha))\,d\alpha, \;\; Am(A) = \int_0^1 (r_A(\alpha) - l_A(\alpha))\,d\alpha$$

are the expected value and ambiguity of the fuzzy number A, respectively. Here $A_\alpha = \{t : \mu_A(t) \ge \alpha\} = [l_A(\alpha), r_A(\alpha)]$ is a α-cut of a fuzzy number (μ_A is a membership function), $\alpha \in (0,1]$.

We will use a linear function $\psi(t) = 1 - \frac{1}{t_0}t$, $t \in [0, t_0]$ at step 2 of the threshold aggregation.

Proposition.

$$v_L(\widetilde{\mathbf{x}}) = \frac{\delta}{1+\delta}\,|L(\widetilde{\mathbf{x}})| - \frac{1}{a(1+\delta)}\sum_{\widetilde{x}_i \in L}(Val(\widetilde{x}_i) + \delta Am(\widetilde{x}_i)),$$

$$v_{M^-}(\widetilde{\mathbf{x}}) = \frac{1+2\delta}{2(1+\delta)}\left|M^-(\widetilde{\mathbf{x}})\right| - \frac{1}{2b(1+\delta)}\sum_{\widetilde{x}_i \in M^-}(Val(\widetilde{x}_i) + \delta Am(\widetilde{x}_i)),$$

$$v_{M^+}(\widetilde{\mathbf{x}}) = \frac{1+2\delta}{2(1+\delta)}\left|M^+(\widetilde{\mathbf{x}})\right| + \frac{1}{2b(1+\delta)}\sum_{\widetilde{x}_i \in M^+}(Val(\widetilde{x}_i) - \delta Am(\widetilde{x}_i)),$$

$$v_H(\widetilde{\mathbf{x}}) = \frac{\delta}{1+\delta}\,|H(\widetilde{\mathbf{x}})| + \frac{1}{a(1+\delta)}\sum_{\widetilde{x}_i \in H}(Val(\widetilde{x}_i) - \delta Am(\widetilde{x}_i)),$$

where $S(\widetilde{\mathbf{x}}) = \{\widetilde{x}_i \in S\}, \;\; S \in \{L, M^-, M^+, H\}$.

Formation of Fuzzy Estimates from Point Data

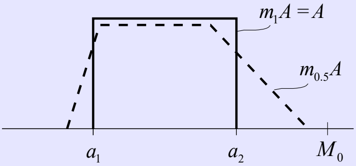
Below we apply this technique to the ranking of crisp data, presented in the form $\{(x_i, \lambda_i)\}_{i=1}^{n}$, where x_i is a point expert estimate (higher value corresponds to higher quality), λ_i is a degree of confidence in the correctness of their decision (higher value corresponds to greater confidence).

Information about the degree of confidence in the correctness of one's decision can be used to blur point estimates. We will adhere to the following **blur principles**:

- the lower the degree of confidence, the greater should be the degree of blurring and ambiguity of fuzzy estimates;
- the degree of blur should be zero for estimates with the highest degree of confidence;
- "extreme" estimates with a low degree of confidence should not move away from the neutral estimate after blurring.

The dependence of blurring on the degree of confidence $\lambda \in [0,1]$ can be modeled using the modifier $m_\lambda : [0,1] \to [0,1]$, $m_\lambda(0) = 0$, $m_\lambda(1) = 1$ according to the rule $\mu_{m_\lambda A}(t) = m_\lambda(\mu_A(t))$. The modifier m_λ must satisfy the conditions:

1) $m_1 A = A$;
2) $Fuz(m_\lambda A) \geq Fuz(m_\tau A)$ for $\lambda \leq \tau$, where Fuz is some (fixed) degree of fuzziness of a fuzzy number;
3) $Am(m_\lambda A) \geq Am(m_\tau A)$ for $\lambda \leq \tau$, where Am is some (fixed) measure of the ambiguity of a fuzzy number;
4) $d(m_\lambda A, M_0) \leq d(m_\tau A, M_0)$ for $\lambda \leq \tau$ and A is an "extreme" fuzzy estimate, i.e. $A \in L$ or $A \in H$.



If A and $m_\lambda A$ are trapezoidal fuzzy numbers and

$$Fuz(A) = |\operatorname{supp} A \backslash \ker A|, \quad Am(A) = \frac{1}{2}\left(|\operatorname{supp} A| + |\ker A|\right),$$

then the conditions 2) and 3) are equivalent to the following

2') $\ker(m_\lambda A) \subseteq \ker(m_\tau A)$ and $\operatorname{supp}(m_\lambda A) \supseteq \operatorname{supp}(m_\tau A)$ for $\lambda \leq \tau$;

3') $|\ker(m_\lambda A)| + |\operatorname{supp}(m_\lambda A)| \geq |\ker(m_\tau A)| + |\operatorname{supp}(m_\tau A)|$ for $\lambda \leq \tau$.

Example

Let nonnegative nonincreasing functions $h_1(\lambda)$ and $h_2(\lambda)$ satisfy the condition $h_1(1) = h_2(1) = 0$. Consider the following external blur modifier

$$m_\lambda A = (a_1 - h_1(\lambda), a_1, a_2, a_2 + h_2(\lambda)).$$

Then we have $Fuz\,(m_\lambda A) = h_1(\lambda) + h_2(\lambda)$ and $Am\,(m_\lambda A) = a_2 - a_1 + \frac{1}{2}\left(h_1(\lambda) + h_2(\lambda)\right)$. This modifier m_λ satisfies conditions 1) – 3).

Numerical Example

Consider an example of ranking articles of conferences in the conference management system, such as EasyChair.

This system uses a septennial scoring system $x_i \in \{-3, -2, -1, 0, 1, 2, 3\}$, corresponding to the recommendations "strong reject", "reject", "weak reject", "borderline paper", "weak accept", "accept", "strong accept". In addition, the reviewer gives an assessment on a five-fold scale (0.2 – "none", 0.4 – "low", 0.6 – "medium", 0.8 – "high", 1 – "expert") about the degree of confidence in the correctness of his decision: $\lambda_i \in \{0.2, 0.4, 0.6, 0.8, 1\}$.

Data $\left\{(x_i^{(k)}, \lambda_i^{(k)})\right\}_{i=1}^{5}$ of $n = 5$ reviewers on 4 articles are presented in Table (i is the index of the reviewer, k is the index of the article, $k = 1, \ldots, 4$), where $L = \{-3, -2, -1\}$, $M = \{0\}$, $H = \{1, 2, 3\}$.

	paper 1	paper 2	paper 3	paper 4
rev. 1	$(2, 0.8)$	$(2, 1)$	$(1, 0.8)$	$(0, 0.6)$
rev. 2	$(1, 1)$	$(2, 0.8)$	$(2, 0.6)$	$(1, 0.4)$
rev. 3	$(0, 0.8)$	$(0, 0.6)$	$(-1, 0.6)$	$(1, 1)$
rev. 4	$(3, 0.4)$	$(-1, 0.4)$	$(0, 0.6)$	$(2, 0.2)$
rev. 5	$(2, 0.6)$	$(1, 0.6)$	$(1, 1)$	$(2, 1)$
$\mathbf{x}^{(k)}$	$(0, 1, 2, 2, 3)$	$(-1, 0, 1, 2, 2)$	$(-1, 0, 1, 1, 2)$	$(0, 1, 1, 2, 2)$
$\mathbf{t}^{(k)}$	(M, H, H, H, H)	(L, M, H, H, H)	(L, M, H, H, H)	(M, H, H, H, H)

If we take into account only the three-grade recommendations of the reviewers ($L = \{-3, -2, -1\}$, $M = \{0\}$, $H = \{1, 2, 3\}$) and do not take into account the degree of confidence, then we will get the vectors of cardinalities $\mathbf{v}(\mathbf{x}^{(k)}) = \big(v_L(\mathbf{x}^{(k)}), v_M(\mathbf{x}^{(k)}), v_H(\mathbf{x}^{(k)})\big)$, $k = 1, \dots, 4$.

If, however, estimates close to the medium ± 1 with a low degree of confidence $\lambda \leq 0.6$ are attributed to the class M, then we will obtain the following (extended) vectors of cardinalities $\mathbf{v}_{\text{ext}}(\mathbf{x}^{(k)})$, $k = 1, \dots, 4$.

	paper 1	paper 2	paper 3	paper 4
$\mathbf{v}(\mathbf{x}^{(k)})$	$(0, 1, 4)$	$(1, 1, 3)$	$(1, 1, 3)$	$(0, 1, 4)$
$\mathbf{v}_{\text{ext}}(\mathbf{x}^{(k)})$	$(0, 1, 4)$	$(0, 3, 2)$	$(0, 2, 3)$	$(0, 2, 3)$
$\mathbf{v}(\widetilde{\mathbf{x}}^{(k)})$	$(0, 0.5, 0.5, 2.59)$	$(0, 1.33, 1.33, 1.3)$	$(0, 1.33, 0.5, 1.57)$	$(0, 0.5, 1.33, 1.75)$

Numerical Example. Fuzzy Aggregation

Let us put each point estimate x in correspondence with the segment: $[0.75(x-1), 0.75x]$ for $x \in \{-3,-2,-1\}$; $[0.75x, 0.75(x+1)]$ for $x \in \{1,2,3\}$; $[-0.75, 0.75]$ for $x = 0$. This segment will be the kernel $\ker(\widetilde{x})$ of the trapezoidal fuzzy number $\widetilde{x}$.

We will use the modifier $m_\lambda A = (a_1 - h_1(\lambda), a_1, a_2, a_2 + h_2(\lambda))$ of the segment-kernel $A = [a_1, a_2]$, where $h_1(\lambda) = 0$, $h_2(\lambda) = \frac{1-\lambda}{5+\lambda}$ for $A \in L$; $h_1(\lambda) = \frac{1-\lambda}{5+\lambda}$, $h_2(\lambda) = 0$ for $A \in H$; $h_1(\lambda) = h_2(\lambda) = \frac{1-\lambda}{5+\lambda}$ for $A \in M$.

Next, we calculate the vectors of fuzzy cardinalities $\mathbf{v}(\widetilde{\mathbf{x}}^{(k)}) = \left(v_L(\widetilde{\mathbf{x}}^{(k)}), v_{M^-}(\widetilde{\mathbf{x}}^{(k)}), v_{M^+}(\widetilde{\mathbf{x}}^{(k)}), v_H(\widetilde{\mathbf{x}}^{(k)})\right)$, $a = 3$, $b = 1.5$, $\delta = 0.3$.

The final rankings obtained by the methods of crisp and fuzzy threshold aggregations are given in Table.

	ranking
$\mathbf{v}(\mathbf{x}^{(k)})$	$\varphi(\mathbf{x}^{(1)}) = \varphi(\mathbf{x}^{(4)}) > \varphi(\mathbf{x}^{(3)}) = \varphi(\mathbf{x}^{(2)})$
$\mathbf{v}_{\text{ext}}(\mathbf{x}^{(k)})$	$\varphi(\mathbf{x}^{(1)}) > \varphi(\mathbf{x}^{(4)}) = \varphi(\mathbf{x}^{(3)}) > \varphi(\mathbf{x}^{(2)})$
$\mathbf{v}(\widetilde{\mathbf{x}}^{(k)})$	$\varphi(\mathbf{x}^{(1)}) > \varphi(\mathbf{x}^{(4)}) > \varphi(\mathbf{x}^{(3)}) > \varphi(\mathbf{x}^{(2)})$

The above example shows that fuzzy threshold aggregation is more sensitive to ranking than the crisp rule. Some alternatives that were indistinguishable with respect to the non-fuzzy threshold aggregation rule began to differ with respect to the fuzzy rule.

Summary and Conclusion

- a procedure for threshold ranking of alternatives represented by vectors of fuzzy numbers has been developed;
- a procedure for blurring point expert data on information about the degree of confidence of experts in their assessments is proposed and investigated;
- the specified procedures of fuzzy threshold aggregation and blurring are demonstrated on the example of ranking articles according to the recommendations of reviewers and the degree of their confidence in their recommendations;
- in the future, it is of interest to develop the axiomatic of fuzzy threshold aggregation.

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Thanks for you attention

alepskiy@hse.ru
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