
Measuring Uncertainty with Imprecision Indices

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Various types of uncertainty

- Randomness (probability theory);
- Nonspecificity (possibility theory);
- Imprecision (interval calculi, monotone measure);
- Inconsistency;
- Incompleteness;
- Fuzziness;

and so on ...

Classical uncertainty measures

- Shannon's entropy (uncertainty – randomness)

$$S(P) = - \sum_{x \in X} P(\{x\}) \log_2 P(\{x\})$$

- Hartley's measure (uncertainty – nonspecificity)

$$H(\eta_{\langle B \rangle}) = \log_2 |B|, \quad \eta_{\langle B \rangle}(A) = \begin{cases} 1, & B \subseteq A, \\ 0, & B \not\subseteq A. \end{cases}$$

Main attributions of particular theory of uncertainty (by G. Klir)

- An uncertainty function g (ex. probability);
- A calculus with functions g ;
- A functional (uncertainty measures) f which measures the amount of uncertainty associated with g (ex. Shannon's entropy);
- A methodology.

Generalized Hartley's measure

Let g be a belief function: $g = \sum_{B \in 2^X} m(B) \eta_{\langle B \rangle}$,

where $m(\emptyset) = 0$, $m(B) \geq 0$, $\sum_{B \in 2^X} m(B) = 1$.

Then $GH(g) = \sum_{B \in 2^X \setminus \{\emptyset\}} m(B) \log_2 |B|$.

Aggregate measure of uncertainty

$$Au(g) = \sup_{P \geq g} S(P)$$

Properties: 1) $Au(P) = S(P),$

2) $Au(\eta_{\langle B \rangle}) = H(\eta_{\langle B \rangle}).$

Basic notations

- M is the set of all set functions on 2^X ;
- $M_0 = \{g \in M \mid g(\emptyset) = 0\}$;
- $M_{mon} = \{g \in M_0 \mid g(X) = 1, g \text{ be monotone measure}\}$;
- M_{Pr} is the set of all probability measures;
- $M_{low} = \{g \in M_{mon} \mid \exists P \in M_{Pr} : g \leq P\}$,
- $M_{up} = \{g \in M_{mon} \mid \exists P \in M_{Pr} : g \geq P\}$;
- M_{bel}, M_{pl} are the sets of all belief and plausibility functions.

Principal motivation

If $g \in M_{low}$ then $\exists P \in M_{Pr} : g \leq P \leq \bar{g}$ where
 $\bar{g}(A) = g(X) - g(\bar{A})$.

Therefore, the “distance” between g and \bar{g}
defines the degree of uncertainty.

Imprecision indices

Let $M = M_{low}$ or $M = M_{up}$.

Definition. A functional $f : M \rightarrow [0,1]$ is called *imprecision index* if :

1) $g \in M_{Pr}$ implies $f(g) = 0$;

2) if $g_1 \leq g_2$ then $f(g_1) \geq f(g_2)$ for $M = M_{low}$
 $(f(g_1) \leq f(g_2) \text{ for } M = M_{up})$;

3) $f(\eta_{\langle X \rangle}) = 1$ for $M = M_{low}$ $(f(\bar{\eta}_{\langle X \rangle}) = 1 \text{ for } M = M_{up})$;

is called *linear imprecision index* ($f \in I(M)$) if extra

4) $f\left(\sum_{j=1}^k \alpha_j g_j\right) = \sum_{j=1}^k \alpha_j f(g_j)$.

Canonical representations of linear imprecision indices

$g = \sum_{B \in 2^X} m_g(B) \eta_{\langle B \rangle} \Rightarrow f(g)$ is defined by

$$f(\eta_{\langle B \rangle}) = \mu_f(B)$$

Then $\mu_f(B) \in M_{mon}$, $\mu_f(\{x\}) = 0 \quad \forall x \in X$.

Let $m^g(B) = \sum_{A: B \subseteq A} (-1)^{|A \setminus B|} g(A)$

be dual Möbius transform.

Proposition. Let f be a linear functional on M
then $f(g) = \sum_{B \in 2^X} m^{\mu_f}(B) g(B)$ for any $g \in M$.

...through description of Möbius transform of μ_f

Theorem. Let f be a linear functional on M .

$$f \in I(M_{low}) \Leftrightarrow \begin{cases} \text{a)} \quad m^{\mu_f}(X) = 1; \quad \sum_{D \in 2^X} m^{\mu_f}(D) = 0; \\ \text{b)} \quad \sum_{D: x \in D} m^{\mu_f}(D) = 0 \quad \text{for all } x \in X; \\ \text{c)} \quad m^{\mu_f}(D) \leq 0 \quad \forall \quad D \in 2^X \setminus \{\emptyset, X\}. \end{cases}$$

Corollary ("avoiding sure loss" condition).

$$f \in I(M_{low}) \Leftrightarrow f(g) = 1 - \sum_{B \in 2^X} m(B)g(B),$$

where: 1) $m(\emptyset) = m(X) = 0$, $m(B) \geq 0 \quad \forall B \in 2^X$;

$$2) \quad \sum_{B \in 2^X} m(B)1_B = 1_X.$$

...through description of μ_f (monotone measure)

Theorem. $f \in I(M_{low}) \Leftrightarrow \mu_f = a\mu - b\bar{\eta}_{\langle X \rangle}$,

where $b > 0$, $a = 1 + b$, $\mu \in M_{pl}$ and with

$$\mu(\{x\}) = b/a \text{ for all } x \in X.$$

Corollary. If $\mu_f = \sum_{A \in 2^X \setminus \{\emptyset\}} m(A) \bar{\eta}_{\langle A \rangle}$ then

$$f \in I(M_{low}) \Leftrightarrow \begin{cases} 1) \quad \mu_f \in M_0(X); \\ 2) \quad \mu_f(\{x\}) = 0 \quad \forall x \in X; \\ 3) \quad m(A) \geq 0 \quad \forall A \in 2^X \setminus \{\emptyset, X\}; \\ \text{or } 3') \quad \mu_f(B \cup \{x\}) \in M_{Pl} \quad \forall x \in X. \end{cases}$$

...through description of distorted function of μ_f

Theorem. Let f be a linear functional on M and

$\mu_f = \lambda \circ P$ and $P(\{x_i\}) = 1/N$, $i = 1, \dots, N$. Then

$$f \in I(M) \Leftrightarrow \begin{cases} \text{a)} \ \lambda(1/N) = 0; \\ \text{b)} \ \lambda \in C^\infty \left[\frac{1}{N}, 1 \right); \\ \text{c)} \ (-1)^{n-1} d^n \lambda(t)/dt^n \geq 0, \ \forall n \in \mathbb{N}, \ t \in \left[\frac{1}{N}, 1 \right). \end{cases}$$

Ex. $\lambda(t) = \ln(t|X|)/\ln(|X|) \Rightarrow$

$\mu_{GH}(A) = \ln(|A|)/\ln(|X|) \Rightarrow GH \in I(M_{low}).$

The algebraic structure of the set of linear imprecision indices

Let $M_I = \{\mu_f : f \in I(M_{low})\}$.

Theorem. Let $\mu \in M_I$, $\mu = \sum_{A \in 2^X \setminus \{\emptyset, X\}} m(A) \bar{\eta}_{\langle A \rangle} - b \bar{\eta}_{\langle X \rangle}$,
for all $A \in 2^X \setminus \{\emptyset, X\}$, $b > 0$, then μ is an extreme point
of $M_I \Leftrightarrow \{1_A\}_{\substack{m(A)>0, \\ A \in 2^X \setminus \{\emptyset, X\}}}^{m(A)>0}$ are linearly independent.

The algebraic structure of the set of complementarily symmetrical imprecision indices

Definition. We call $f \in I(M_{low})$ *complementarily symmetrical* if $m^{\mu_f}(A) = m^{\mu_f}(\bar{A}) \quad \forall A \in 2^X \setminus \{\emptyset, X\}$.
Ex. Let $\nu_B(g) = g(X) - g(B) - g(\bar{B}) + g(\emptyset)$ be primitive imprecision index. Then

$$\mu_{\nu_B}(A) = \bar{\eta}_{\langle B \rangle}(A) + \bar{\eta}_{\langle \bar{B} \rangle}(A) - \bar{\eta}_{\langle X \rangle}(A) \Rightarrow (\text{such as } \{1_B, 1_{\bar{B}}\} \text{ are linearly independent}) \Rightarrow \nu_B \text{ - extreme point} \Rightarrow$$

Theorem. Let f be complementarily symmetrical $\Leftrightarrow f = \sum_B \alpha(B) \nu_B, \sum_B \alpha(B) = 1, \alpha(B) \geq 0 \quad \forall B$.

The extension of imprecision indices to the set

$$M_{low} \cup M_{up}$$

Let $\bar{f}(g) = f(\bar{g})$.

Proposition. $f \in I(M_{low}) \Leftrightarrow \bar{f} \in I(M_{up})$.

Proposition. Let f be a linear functional on M

then $|f| \in I(M_{low} \cup M_{up}) \Leftrightarrow f$ is
complementarily symmetrical index on M_{low} .

The extension of imprecision indices to the set of all monotone measures M_{mon}

uncertainty = imprecision \cup inconsistency

Let $f \in I(M_{low})$, $g \in M_{mon}$ but $g \notin M_{low}$ then

$f_{Imp}(g) = \inf_{q \in M_{low} | q \leq g} f(q)$ is the **amount of imprecision** in g.

If $g \in M_{up}$ then $f_{Imp}(g) = 0 \Rightarrow \{\text{imprecision}\} = 0$,

$\{\text{uncertainty}\} = \{\text{inconsistency}\}$.

By analogy, if $g \in M_{mon}$ but $g \notin M_{up}$ then

$f_{Inc}(g) = \inf_{q \in M_{up} | q \geq g} f(\bar{q})$ is **amount of inconsistency** in g.

If $g \in M_{low}$ then $f_{Inc}(g) = 0 \Rightarrow \{\text{inconsistency}\} = 0$,

$\{\text{uncertainty}\} = \{\text{imprecision}\}$.

Properties of indices on M_{mon}

- 1) $g_1 \geq g_2 \Rightarrow f_{\text{Imp}}(g_1) \geq f_{\text{Imp}}(g_2), f_{\text{Imp}}(g_1) \leq f_{\text{Imp}}(g_2);$
- 2) $f_{\text{Imp}}(g) = \inf_{\alpha \in M_{\text{Pr}}} f(\min\{\alpha, g\}),$
 $f_{\text{Inc}}(g) = \inf_{\alpha \in M_{\text{Pr}}} f(\min\{\alpha, \bar{g}\});$
- 3) $g \in M_{mon}$ is rather lower probability than upper probability if $f_{\text{Imp}}(g) \geq f_{\text{Inc}}(g)$ and rather upper probability than lower probability if $f_{\text{Imp}}(g) < f_{\text{Inc}}(g);$
- 4) if f – complementarily symmetrical index on M_{low} then $f_{\text{Imp}} - f_{\text{Inc}} = f.$

Example

X	x_1	x_2	x_3
π_1	1	0.5	0.5
π_2	0.4	1	0.6

$$\begin{aligned}\Pi_i(A) &= \max_{x \in X} \pi_i(x) \in M_{up}, \\ N_i(A) &= 1 - \Pi_i(\bar{A}) \in M_{low}, \\ g &= \max\{N_1, N_2\} \notin M_{low}.\end{aligned}$$

Let $v_1(g) = (2^{|X|} - 2)^{-1} \sum_{B \in 2^X} |\bar{g}(B) - g(B)|$,

$$v_\infty(g) = \max \left\{ |\bar{g}(B) - g(B)| \mid B \in 2^X \right\}.$$

	f_{Imp}			f_{Inc}		
	v_1	v_∞	GH	v_1	v_2	GH
N_1	0.5	0.5	0.5	0	0	0
N_2	0.5(3)	0.6	0.526	0	0	0
g	0.2	0.5	0.2	0.03(3)	0.1	0.0288