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ScienceDirect

Procedia Computer Science 122 (2017) 186–193

Procedia Computer Science

www.elsevier.com/locate/procedia

Information Technology and Quantitative Management (ITQM 2017)

Decomposition of Evidence and Internal Conflict

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Abstract

This paper is devoted to the study of estimation of internal conflict of evidence in the framework of belief functions theory. The decomposition approach, which was proposed early by the author, will be considered with this purpose. The establishment of some properties of this conflict measure is a main result of this paper. In particular, estimates of the upper bound of the internal conflict in the case of categorical evidence and bifocal bodies of evidence are obtained for decomposition with the help of Dempster's rule and Dubois and Prade's disjunctive rule.

Peer-review under responsibility of the scientific committee of the 5th International Conference on Information Technology and Quantitative Management, ITQM 2017.

Keywords: conflict management, belief functions theory, internal conflict, combining rules

1. Introduction

Aggregation of information sources (information fusion) is the important direction in information processing that used in various applied fields, such as image processing [13], multi sensor fusion [22], aggregation of experts information in economic and financial analysis [3], pattern recognition [7], etc. Different approaches can be used for aggregation of information. The theory of evidence (the belief functions theory, the Dempster-Shafer theory [6, 24]) is a convenient tool for aggregating information. The theory of evidence is a convenient tool for aggregating information because it has an advanced tool of combining rules. These rules establish the correspondence between of two (or more) bodies of evidence (belief functions) and one body of evidence (belief function). The plausibility of applying the combining rules depends on a number of features of the bodies of evidence. The conflict between bodies of evidence [8, 17], the reliability of information sources [24], the uncertainty of evidence [2, 9] are most important examples of such features.

The study of conflict between two (or more) bodies of evidence (we will call this type of conflict an external conflict) took shape in an independent scientific direction recently [3, 4, 8, 14, 17, 20, 21]. The external conflict characterizes the degree of difference both between the sets of focal elements of two bodies of evidence and between the mass functions defined on the set of focal elements. For example, two statements the value of the company shares will be tomorrow in the interval $[30, 40]$ and the value of the company shares will be tomorrow in the interval $[50, 60]$ (and corresponding bodies of evidence with focal elements $A = [30, 40]$ and $B = [50, 60]$) have a big external conflict because $A \cap B = \emptyset$. The conflict related to Dempster's combining rule [6] is the first

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Peer-review under responsibility of the scientific committee of the 5th International Conference on Information Technology and Quantitative Management, ITQM 2017.
10.1016/j.procs.2017.11.359

conflict measure. This conflict measure (we will call this measure a canonical conflict measure) is a generalization of notion of covariance for probability distributions. The notion of conflict between pieces of evidence is widely used in deciding on the choice of information sources for combining in various conflict management tasks [16].

However there is considered also the internal conflict of single body of evidence. For example, the statement the value of the company shares will be tomorrow in the intervals [30,40] or [60,80] has a big internal conflict. There are a few approaches to estimation of internal conflict: entropy approach, approach based on the plausibility function, decomposition approach. These approaches will be consider below.

In [19] was suggested new decomposition approach to estimation of internal conflict. This approach proposes that conflict is a result of a some combining (with the help of unknown rule) of a some set of bodies of evidence. Then we can estimate the internal conflict of evidence with the help of a some measure of external conflict applied to bodies of evidence that was received by decomposition of initial evidence. In [19] was considered the general setting of a problem of estimation of internal conflict with the help of decomposition of evidence. The decomposition problem has a many of solutions in general. Therefore the some restrictions must be considered for getting of plausible set of solutions. These restrictions was considered in [19] also as a some particular cases.

The present paper continue the study of estimation of internal conflict with the help of decomposition of evidence. In particular, the general properties of such internal conflict will establish. The rest of the paper has the following structure. Section 2 presents basic notions of the belief functions theory and the conflict measures. Some approaches to estimation of internal conflict are described in Section 3. Evaluation of the internal conflict with the help of decomposition is given in Section 4. Estimates of the upper bound of the internal conflict in the case of categorical evidence and bifocal bodies of evidence are found in Section 5 for decomposition with the help of Dempster's rule and in Section 6 for decomposition with the help of Dubois and Prade's disjunctive rule. Finally, Section 7 presents some conclusions.

2. Background of the Belief Functions Theory and Conflict Measures

Let X be a finite set and 2^X be a powerset of X. The mass function is a set function $m : 2^X \to [0,1]$ such that $m(\emptyset) = 0$, $\sum_{A\subseteq X} m(A) = 1$. The value $m(A)$ characterizes the degree of belonging of the actual alternative from X to the set $A \in 2^X$.

A subset $A \in 2^X$ is called a focal element, if $m(A) > 0$. Let $\mathcal{A} = \{A\}$ be a set of all focal elements. The pair $F = (\mathcal{A}, m)$ is called a body of evidence. Let $F_A = (A, 1)$ (i.e. $\mathcal{A} = \{A\}$ and $m(A) = 1$), $A \in 2^X$, then it is called a categorical evidence. In particular, $F_X = (X, 1)$ is called vacuous evidence because the body of evidence $F_X = (X, 1)$ is totally uninformative. Let $\mathcal{F}(X)$ be a set of all possible bodies of evidence on X.

If $F_j = (\mathcal{A}_j, m_j) \in \mathcal{F}(X)$, $j = 1, ..., n$ and $\sum_{j=1}^n \alpha_j = 1$, $0 \le \alpha_j \le 1$, $j = 1, ..., n$, then $F = (\mathcal{A}, m) \in \mathcal{F}(X)$, where $\mathcal{A} = \bigcup_{j=1}^n \mathcal{A}_j$, $m(A) = \sum_{j=1}^n \alpha_j m_j(A)$. In this case, we will write $F = \sum_{j=1}^n \alpha_j F_j$. In particular, any body of evidence $F = (\mathcal{A}, m)$ can be represented as $F = \sum_{A\in\mathcal{A}} m(A) F_A$.

If $F = (\mathcal{A}, m)$ is body of evidence then a so called belief function [24] $g : 2^X \to [0,1]$, $g(B) = \sum_{A\subseteq B} m(A)$ characterizes the belief degree that the true alternative of X belongs to set B. The dual measure $Pl(B) = 1 - Bel(\neg B) = \sum_{A\cap B\neq\emptyset} m(A)$ is called by plausibility function. The bodies of evidence, belief and plausibility functions can be defined on non-finite set (for example, on n-dimensional Euclidean space). In last case we will propose that the set of focal elements $\mathcal{A}$ is a finite set.

Let us have two bodies of evidence $F_1 = (\mathcal{A}_1, m_1)$ and $F_2 = (\mathcal{A}_2, m_2)$ that were obtained from two information sources. Then the combining rule $R : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathcal{F}(X)$ is considered. Dempster's rule was combining rule and it is defined as $m_D(A) = \frac{1}{1-K_0} \sum_{B\cap C=A} m_1(B)m_2(C), \quad A \neq \emptyset, \quad m_D(\emptyset) = 0$, where

$$K_0 = K_0(F_1, F_2) = \sum_{B\cap C=\emptyset, B\in\mathcal{A}_1, C\in\mathcal{A}_2} m_1(B)m_2(C). \tag{1}$$

The value $K_0(F_1, F_2)$, that we called by the canonical conflict measure, characterizes the amount of conflict between two sources of information described by the bodies of evidence F_1 and F_2. If $K_0 = 1$ (this situation corresponds to absolute conflict: $A \cap B = \emptyset$ for all $A \in \mathcal{A}_1$, $B \in \mathcal{A}_2$), then Dempster's rule is not applied.

Dubois and Prade's disjunctive consensus rule [10] is a dual rule to Dempster's rule in some sense. This rule is defined by

$$m_{DP}(A) = \sum_{B\cup C=A} m_1(B)m_2(C), \quad A \in 2^X. \tag{2}$$

Other combining rules see, for example, in [18, 25].

3. Some Approaches to Estimation of Internal Conflict

There are a few approaches to estimation of internal conflict. For example, we can highlight the following approaches.

1. Entropy approach. This a historically the first approach that considered in the early 1980s. In this case the conflict was estimated with the help of a some functional that was generalization of Shannon entropy. The following functional were considered: dissonance, confusion, discord, strife etc. [15]. For example, measure of confusion [12] is defined as a mean of conflict focal elements $C(F) = -\sum_{A\in\mathcal{A}} m(A)\log_2(1 - L(A))$ with respect to relation of "non-inclusion" $L(A) = \sum_{B\not\subseteq A} m(B)$.

2. Axiomatic approach. This approach was considered in [1, 11].

3. Approach based on the plausibility function. This approach is based on the following observation. The body of evidence $F = (\mathcal{A}, m)$ can be considered internally non-conflicting if it does not conflict with itself with respect to the canonical measure of the conflict, i.e. $K_0(F, F) = 0$. It means that $\forall A, B \in \mathcal{A}$: $A \cap B \neq \emptyset$. Consequently, we have $\Omega_{\mathcal{A}} = \bigcap_{A\in\mathcal{A}} A \neq \emptyset$. Therefore $Pl(x) = 1$ $\forall x \in \Omega_{\mathcal{A}}$ if X be a finite set. A measure of internal conflict was introduced in [5], based on this observation, as $K_{in}(F) = 1 - \max\{Pl(x) : x \in X\}$. But this approach is difficult to generalize to the case of an uncountable set X.

4. Decompositional Approach. This approach means that initial body of evidence must be decomposed into the some set of bodies of evidence. Then the external conflict of this set of bodies of evidence is estimated with the help of a some conflict measure. In [23] (see also [22]) the initial body of evidence decomposes uniquely into the set of so called generalized simple support functions. But this approach has a two disadvantages. First, not every body of evidence can be decomposed into the set of generalized simple support functions. Second, the canonical decomposition is not related directly to the possible combining rule.

Therefore we will use the other decompositional approach that was introduced in [19]. This approach assumes that initial body of evidence $F = (\mathcal{A}, m)$ can be considered as a result of combining several bodies of evidence $F_i = (\mathcal{A}_i, m_i) \in \mathcal{F}(X)$, $i = 1, ..., l$ with the help of some combining rule R: $F = R(F_1, ..., F_l)$. Therefore we can estimate the internal conflict by the formula $K_{in}(F) = K(F_1, ..., F_l)$ assuming that $F = R(F_1, ..., F_l)$, where K is some fixed (external) conflict measure, R is a fixed combining rule. Since the equation $F = R(F_1, ..., F_l)$ has many solutions we can consider the optimization problem of finding the largest $\underline{K}_{in}^R(F)$ and smallest $\overline{K}_{in}^R(F)$ conflicts:

$$\underline{K}_{in}^R(F) = \sup_{F=R(F_1,...,F_l)} K(F_1, ..., F_l), \quad \overline{K}_{in}^R(F) = \inf_{F=R(F_1,...,F_l)} K(F_1, ..., F_l). \tag{3}$$

Let $S_n = \left\{\mathbf{s} = (s_i)_{i=1}^n : s_i \geq 0 \ \forall i = 1, ..., n, \ \sum_{i=1}^n s_i = 1\right\}$ be a n-dimensional simplex.

Decomposition of evidence with the help of Dempster's rule. Let $R = D$ be the Dempster's rule. Then optimization problems (3) for $l = 2$ have the following formulation. We have to find

$$K_0(F_1, F_2) = \sum_{B\cap C=\emptyset, B\in\mathcal{A}_1, C\in\mathcal{A}_2} m_1(B)m_2(C) \to \sup \quad (\inf) \tag{4}$$

with constraints

$$\mathbf{m}_1 = (m_1(B))_{B\in\mathcal{A}_1} \in S_{|\mathcal{A}_1|}, \quad \mathbf{m}_2 = (m_2(C))_{C\in\mathcal{A}_2} \in S_{|\mathcal{A}_2|}, \tag{5}$$

$$(1 - K_0(F_1, F_2))\, m(A) = \sum_{B\cap C=A, B\in\mathcal{A}_1, C\in\mathcal{A}_2} m_1(B)m_2(C), \quad A \in \mathcal{A}, \tag{6}$$

$$K_0(F_1, F_2) < 1. \tag{7}$$

In the case of the general formulation (4)-(7) $\underline{K}_{in}^{D}(F) = 0$ and this value is achieved on the pair bodies of evidence $F_1 = F$ and $F_2 = F_X$. We can also observe that largest value of conflict measure for two bodies of evidence satisfying conditions (5)-(6) (without (7)) is equal to $K_0(F_1, F_2) = 1$, and this value is achieved for such $F_i = (\mathcal{A}_i, m_i) \in \mathcal{F}(X)$, $i = 1, 2$, that $B \cap C = \emptyset \quad \forall B \in \mathcal{A}_1,\ \forall C \in \mathcal{A}_2$.

Decomposition of evidence with the help of Dubois and Prade's disjunctive consensus rule. Let $R = DP$ be a Dubois and Prade's disjunctive consensus rule (2). Then the conditions (2) will be used instead of the conditions (6) in the problem of finding the internal conflict. Thus, in this case we have the problem of finding of bodies of evidence of having the largest (smallest) canonical conflict (4) and satisfying the conditions (2), (5).

It is convenient to consider that the empty set can also be a focal element of evidence in the case of using Dubois and Prade's disjunctive consensus rule. This can be interpreted as $x \notin X$ and a value $m(\emptyset)$ characterizes the degree of belief to the fact $x \notin X$. Then the largest value of conflict measure (4) satisfying conditions (2), (5) will be equal $\overline{K}_{in}^{DP}(F) = 1$. This value is achieved for the following decomposition body of evidence F: $F_1 = F$, $F_2 = F_\emptyset$. The solution of decomposition problem in the case when the empty set can be focal elements we will call the generalized decomposition solution and we will denote the corresponding estimates as $\underline{\mathcal{K}}_{in}^{DP}(F)$ and $\overline{\mathcal{K}}_{in}^{DP}(F)$.

We see that in general the problem of finding $\overline{K}_{in}^{D}(F)$ and $\underline{K}_{in}^{D}(F)$ may have trivial solutions. But there are a some restrictions that conditioned by the character of combining rule. For example, Dempster's rule is an optimistic rule in the following sense. If one evidence argues that a true alternative belongs to the set A, and the second certifies that it is in the set B, then after combination of pieces of evidence by the Dempster's rule we get that the true alternative is in the set $A \cap B$ (see [18]). These limitations can be taken into account with the help of so called imprecision index [2].

Definition 1. *A functional $f : \mathcal{F}(X) \to [0, 1]$ is called imprecision index if the following conditions are fulfilled:*
1. *if $F = (\mathcal{A}, m) \in \mathcal{F}(X)$: $|A| = 1$ for all $A \in \mathcal{A}$ (i.e. $F \leftrightarrow g$, where g is a probability measure), then $f(F) = 0$;*
2. *$f(F_1) \geq f(F_2)$ for all $F_1, F_2 \in \mathcal{F}(X)$ such that $F_i \leftrightarrow g_i$, $i = 1, 2$ and $g_1 \leq g_2$ (i.e. $g_1(A) \leq g_2(A)$ for all $A \subseteq X$);*
3. *$f(F_X) = 1$.*

An imprecision index is called strict if: 1) $f(F) = 0 \Leftrightarrow F \leftrightarrow g$, where g is a probability measure; 2) $f(F) = 1 \Leftrightarrow F = F_X$.

An imprecision index f on $\mathcal{F}(X)$ is called linear if we have $f\left(\sum_{j=1}^{n} \alpha_j F_j\right) = \sum_{j=1}^{n} \alpha_j f\left(F_j\right)$ for any $F_j \in \mathcal{F}(X)$, $j = 1, ..., n$ and $(\alpha_j) \in S_n$.

The value $f(F)$ characterizes the amount of ignorance in the information contained in evidence F. Different descriptions of imprecision indices see in [2]. In this paper we will use the following representation.

Proposition 1. *A functional $f : \mathcal{F}(X) \to [0, 1]$ is a linear imprecision index on $\mathcal{F}(X)$ iff $f(F) = \sum_{B \in \mathcal{A}} m(B)\mu_f(B)$, $F = (\mathcal{A}, m)$, where monotone set function $\mu_f(B) = f(F_B)$, $B \neq \emptyset$, $\mu_f(\emptyset) = 0$ satisfies the conditions: 1) $\mu_f(\{x\}) = 0$ for all $x \in X$; 2) $\mu_f(X) = f(F_X) = 1$; 3) $\sum_{B: A \subseteq B} (-1)^{|B \setminus A|} \mu_f(B) \leq 0$ for all $A \neq \emptyset, X$.*

Normalized generalized Hartley's measure [9] $f(F) = \sum_{A \in \mathcal{A}} m(A) \log_{|X|} |A|$ is an example of a strict linear imprecision index.

Let us have two sources of information, and this information is described by categorical bodies of evidence F_A and F_B respectively, where $A, B \in 2^X \backslash \{\emptyset\}$. The first source states that true alternative is contained in the set A, and the second source states that true alternative is contained in the set B.

If we apply Dubois and Prade's disjunctive consensus rule for these categorical bodies of evidence then we get $DP(F_A, F_B) = F_{A \cup B}$. In other words, we get the statement that a true alternative is contained in the set $A \cup B$. This statement can be considered as more pessimistic than an initial statement because uncertainty does not decrease after combining. For example, if linear imprecision index of initial bodies of evidence is equal to $f(F_A) = \mu_f(A)$ and $f(F_B) = \mu_f(B)$ respectively, then this index is equal to $f(F_{A \cup B}) = \mu_f(A \cup B) \geq f(F_A)$ for the resulting evidence.

If we apply Dempster's rule to these categorical bodies of evidence then we get $D(F_A, F_B) = F_{A \cap B}$ for $A \cap B \neq \emptyset$ (if $A \cap B = \emptyset$, then $K = 1$ and Dempster's rule cannot be applied). After combining we got the statement that a true alternative is contained in the set $A \cap B$. This statement can be considered as more optimistic than an initial statement because uncertainty does not increase after combining: $f(F_{A\cap B}) = \mu_f(A \cap B) \le f(F_A)$.

The following propositions describe the different character of combining rule (see for details [18]).

Proposition 2. *If $F = DP(F_1, F_2)$, $F_1, F_2 \in \mathcal{F}(X)$, where DP is Dubois and Prade's disjunctive consensus rule, then inequalities $f(F) \ge f(F_i)$, $i = 1, 2$ are true for any linear imprecision index f.*

Proposition 2 shows us that information uncertainty cannot be decreased if we aggregate information from many sources with the help of Dubois and Prade's disjunctive consensus rule.

Proposition 3. *Let F_1, F_2 are such bodies of evidence that their conflict measure $K_0(F_1, F_2) = 0$ and $F = D(F_1, F_2)$, where D be Dempster's rule. Then inequalities $f(F) \le f(F_i)$, $i = 1, 2$ are true for any linear imprecision index f.*

4. Evaluation of the Internal Conflict by Decomposition Method

Let we consider the decomposition on only two bodies of evidence with the help Dempsters rule. Then we have the following formulation of this problem. We have to find the largest and smallest values of conflict measure $K_0(F_1, F_2)$ on restrictions (5)-(7) and conditions:

$$f(F) \le f(F_i), \quad i = 1, 2, \tag{8}$$

where $f : \mathcal{F}(X) \to [0, 1]$ is an imprecision index. We will use restrictions (8) without assumption about absence of conflict between bodies of evidence F_i, $i = 1, 2$. Conditions (8) are performed for the bodies of evidence $F_1 = F$ and $F_2 = F_X$ since $f(F_X) = 1$. Therefore we always have $\underline{K}_{in}^D(F) = 0$. Then the problem can be put to find bodies of evidence with the largest conflict (4) and satisfying the conditions (5)-(7), (8).

The upper limitations on the amount of ignorance in the information contained in the decomposable bodies of evidence can be seen also: $f(F_i) \le f_{\max}$, $i = 1, 2$, where $f_{\max}$ is a maximum allowable level of ignorance.

If the decomposition of evidence F is performed with the help of Dubois and Prade's disjunctive consensus rule then we will use condition (2) instead of conditions (6) in the problem of finding of internal conflict. In addition (see Proposition 2), the following estimation holds for Dubois and Prade's disjunctive consensus rule and any linear imprecision index f:

$$f(F) \ge f(F_i), \quad i = 1, 2, \tag{9}$$

i.e. imprecision of evidence is not reduced after the application of this combining rule.

Thus, we have a problem of finding evidences having the largest (smallest) conflict (4) and satisfying constraints (2), (5), (9).

We have the following presentation of internal conflict in the case $|X| = 2$. If decompositions based on Dempster's rule then we have the following result [19].

Proposition 4. *Let $F = m_0F_X + m_1F_{\{x_1\}} + m_2F_{\{x_2\}} \in \mathcal{F}(X)$, $|X| = 2$. Then*

$$\overline{K}_{in}^D(F) = \frac{m_1m_2}{(1 - m_1)(1 - m_2)} = \frac{m_1m_2}{(m_0 + m_1)(m_0 + m_2)} \quad \textit{if} \;\; m_0 \ne 0$$

and $\overline{K}_{in}^D(F) = 1$ if $m_0 = 0$.

If decompositions based on Dubois and Prade's disjunctive rule then we have the following result [19].

Proposition 5. *Let $F = m_0F_X + m_1F_{\{x_1\}} + m_2F_{\{x_2\}} \in \mathcal{F}(X)$, $|X| = 2$ and $m_1 \ne 0$, $m_2 \ne 0$. Then the decomposition problem (2), (4, (5), (8) has a solution iff $\sqrt{m_1} + \sqrt{m_2} \le 1$ and in this case we have*

$$\underline{K}_{in}^{DP}(F) = 2\sqrt{m_1m_2}, \;\; \overline{K}_{in}^{DP}(F) = m_0 = 1 - m_1 - m_2.$$

It is easy to show also that $\overline{K}_{in}^{D}(F) \le \underline{K}_{in}^{DP}(F)$ for all $F = m_0 F_X + m_1 F_{\{x_1\}} + m_2 F_{\{x_2\}} : \sqrt{m_1} + \sqrt{m_2} \le 1$. This means that the estimation of internal conflict obtained with the help of optimistic Dempster's rule is never greater than the estimation of an internal conflict obtained with the help of pessimistic Dubois and Prade's disjunctive consensus rule.

We will establish some properties of the internal conflict, calculated with the help of decomposition by Dempster's rule.

5. Some Properties of the Internal Conflict with the Help of Dempster's Rule

5.1. Internal Conflict of Categorical Evidence

We remind that $\underline{K}_{in}^{D}(F) = 0$ for every $F = (\mathcal{A}, m) \in \mathcal{F}(X)$. The following proposition shows us that upper bound of internal conflict will be zero also for vacuous evidence $F_X = (X, 1)$.

Proposition 6. $\underline{K}_{in}^{D}(F_X) = \overline{K}_{in}^{D}(F_X) = 0$ *for any strict imprecise index f.*

If we consider an arbitrary categorical body of evidence $F_A = (A, 1)$, $A \in 2^X \backslash \{\emptyset, X\}$ then upper bound is not equal to zero in general case. In particular, the following proposition holds.

Proposition 7. *Let $A \in 2^X \backslash \{\emptyset, X\}$. Then the following estimation is true for linear imprecision index f:*

$$\overline{K}_{in}^{D}(F_A) \ge \sup\left\{\sum\nolimits_{j=1}^{m} \beta_j\right\},$$

where supremum is taken over all $(\beta_j)_1^m$: $0 \le \beta_j \le 1$, $j = 1, ..., m$, $\sum_{j=1}^{m} \beta_j < 1$, satisfying inequality $\sum_{j=1}^{m} \beta_j (1 - \mu_f(B_j)) \le 1 - \mu_f(A)$ for some system of subsets $\{B_j\}_{j=1}^{m}$, for which $A \cap B_j = \emptyset \; \forall j = 1, ..., m$.

Corollary 1. *If $A \in 2^X \backslash \{\emptyset, X\}$, then the following estimation is true for linear imprecision index f*

$$\overline{K}_{in}^{D}(F_A) \ge \sup\left\{\beta \in [0, 1) : \; \beta(1 - \mu_f(X \backslash A)) \le 1 - \mu_f(A)\right\}.$$

Remark 1. *The latest estimation for the upper bound of the internal conflict can be written as*

$$\overline{K}_{in}^{D}(F_A) \ge \min\left\{1, \frac{1 - \mu_f(A)}{1 - \mu_f(X \backslash A)}\right\},$$

if we assume that $\frac{0}{0} = 1$.

Corollary 2. *If $A \in 2^X \backslash \{\emptyset, X\}$ satisfies the condition $\mu_f(A) \le \mu_f(X \backslash A))$ for linear imprecision index f then $\overline{K}_{in}^{D}(F_A) = 1$.*

The value $\mu_f(A)$ characterizes the relative "volume" of the set A. The Corollary 2 shows us that largest upper bound (equal to 1) of internal conflict will be in the case when focal set has a small "volume" with respect to "volume" of its complement. For example, let $X = [0, 100]$ is a segment of the predictive value of the shares of a company. Let us have body of evidence $F_A = ([50, 55], 1)$, i.e. the source of information tells us that value of the company shares will be in the segment $A = [50, 55]$. The focal set $A = [50, 55]$ is a "small" relative to the set $X \backslash A$ with respect to (for example) $\mu_f(A) = \ln|A| / \ln|X|$, where $|[a, b]| = b - a + 1$. This body of evidence could be obtained by combining of bodies of evidence $F_1 = ([0, 50), [50, 55]; 1 - \varepsilon, \varepsilon)$ and $F_2 = ([50, 55], (55, 100]; \varepsilon, 1 - \varepsilon)$ with the help of Dempster's rule: $F_A = D(F_1, F_2)$. The conditions (8) are fulfilled and $K_0(F_1, F_2) = (1 - \varepsilon)^2$. If the focal set A is a "large" relative to the set $X \backslash A$, then the body of evidence F_A cannot be obtained analogously to the performance of condition (8).

In particular, Corollary 2 implies that the internal conflict of Dirac's measure has the greatest uncertainty.

Corollary 3. $\overline{K}_{in}^{D}(F_{\{x\}}) = 1$ *for every $x \in X$.*

5.2. The Internal Conflict of Bifocal Evidence

Let us now study the internal conflict of the bifocal evidence, i.e. evidence of the form $F = (\mathcal{A}, m)$, $\mathcal{A} = \{A, B\}$. We consider first the case when the focal elements are disjoint: $A \cap B = \emptyset$.

Proposition 8. *If $F = mF_A + (1 - m)F_B$, $A \cap B = \emptyset$, $m \in (0, 1)$, then $\overline{K}_{in}^{D}(F) = 1$ for linear imprecision index f.*

Proposition 8 shows us that bifocal evidence with disjoint focal elements has the maximal potential conflict (i.e. $\overline{K}_{in}^{D}(F) = 1$).

The situation is more complex for bifocal evidence with intersecting focal elements. See examples below.

Example 1. *Let $X = \{x_1, x_2, x_3\}$ and $F = mF_{\{x_1,x_2\}} + (1 - m)F_{\{x_2,x_3\}}$, $m \in (0, 1)$. Then $\overline{K}_{in}^{D}(F) = 0$.*

Example 2. *Let $X = \{x_1, x_2, x_3\}$ and $F = mF_{\{x_2\}} + (1-m)F_{\{x_2,x_3\}}$, $m \in (0, 1)$. Then $\overline{K}_{in}^{D}(F) = 1-(1-m)\mu_f(\{x_2, x_3\})$ and this value is achieved, for example, for bodies of evidence $F_1 = F$ and $F_2 = \left(1 - (1 - m)\mu_f(\{x_2, x_3\})\right) F_{\{x_1\}} + (1 - m)\mu_f(\{x_2, x_3\})F_X$.*

6. Some Properties of the Internal Conflict with the Help of Dubois and Prade's Disjunctive Rule

We can prove the following properties of the internal conflict estimated with the help of Dubois and Prade's disjunctive rule by analogically with previous section.

Proposition 9. *If $A \in 2^X \backslash \{\emptyset\}$, then $\underline{K}_{in}^{DP}(F_A) = 0$. If in addition $|A| > 1$ then $\overline{K}_{in}^{DP}(F_A) = 1$.*

In the case of generalized solutions of the decomposition problem we have the following property.

Proposition 10. *$\underline{\mathcal{K}}_{in}^{DP}(F_\emptyset) = 1$ for strict imprecision index f.*

Estimates of the internal conflict of Dirac's measure are different for simple and generalized solutions of the decomposition problem.

Proposition 11. *The following properties are true for strict imprecision index f:*
1) $\overline{K}_{in}^{DP}(F_{\{x\}}) = 0$ for every $x \in X$;
2) $\underline{\mathcal{K}}_{in}^{DP}(F_{\{x\}}) = 0$, $\overline{\mathcal{K}}_{in}^{DP}(F_{\{x\}}) = 1$ for every $x \in X$

Finally, we will establish some properties for bifocal evidence.

Proposition 12. *Let $A, B \in 2^X \backslash \{\emptyset\}$, $A \cap B \neq \emptyset$, $m \in (0, 1)$. Then we have:*
1) $\underline{K}_{in}^{DP}(mF_A + (1 - m)F_B) = 0$;
2) If, in addition $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$, then $\overline{K}_{in}^{DP}(mF_A + (1 - m)F_B) = 1$.

Example 3. *Let $X = \{x_1, x_2\}$ and $F = mF_{\{x_1\}} + (1 - m)F_{\{x_1,x_2\}}$, $m \in (0, 1)$. Then (see Proposition 5) $\underline{K}_{in}^{DP}(F) = 0$ and $\overline{K}_{in}^{DP}(F) = 1 - m$. This example shows us that if additional conditions in Proposition 12 are not true, then upper bound of internal conflict may be differ from zero.*

7. Conclusions

This paper continues the study of a new approach to estimation of the internal conflict based on decomposition of initial evidence with a some reasonable restrictions. The main purpose of this article was to establish simple but important theoretical properties. In particular, estimates were found for upper bound of the internal conflict in the case of categorical evidence and bifocal bodies of evidence in the case of decomposition with the help of Dempster's rule and Dubois and Prade's disjunctive rule.

In the case of Dempster's rule it is shown that:

- the internal conflict of vacuous evidence is the minimum (zero);

- conditions are found for which internal conflict of categorical evidence has a maximal uncertainty;
- bifocal evidence with disjoint focal elements has the maximum potential conflict.

In the case of Dubois and Prade's disjunctive rule it is shown that:

- the internal conflict of Dirac's measure is the minimum (zero);
- the generalized internal conflict of empty evidence is the maximum (one);
- categorical evidence with cardinality of focal element greater than one has the maximum potential conflict;
- bifocal evidence with intersecting but not containing each other focal elements has the maximum potential conflict.

These properties show us that internal conflict estimates based on decomposition method are plausible and it can be used in applications.

Acknowledgments.

The financial support from the Government of the Russian Federation within the framework of the implementation of the 5-100 Programme Roadmap of the National Research University Higher School of Economics is acknowledged.

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