Application of Fuzzy Asymmetric GARCH-Models to Forecasting of Volatility of Russian Stock Market

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Abstract. This paper presents the results of volatility forecasting for indices of the Russian stock market using existing and developed by the authors fuzzy asymmetric GARCH-models. These models consider various switching functions which are taking into account the positive and negative shocks and are built using the tools of fuzzy numbers. Furthermore, in some models there are used switching functions that consider expert macroeconomic information. It was shown that fuzzy asymmetric GARCH-models provide a more accurate prediction of volatility than similar crisp models.

Keywords: volatility, asymmetric GARCH-model, fuzzy numbers

1 Introduction

Volatility is one of the key risk parameters in the finance theory. Therefore, the problem of volatility forecasting is one of the central tasks of financial analysis [11]. Successful financial risk management, pricing of options and many other things depend on the accuracy of this problem solution.

Under the efficient market hypothesis, models of autoregressive conditional heteroscedasticity – ARCH [3] and GARCH [2] are widely used to predict volatility. These models have many modifications. Especially, asymmetric GARCH-models are popular because they differently take into account negative and positive shocks which gives a more accurate prediction on asymmetric data. Such models are, for example, Threshold GARCH – TGARCH [12] and Volatility Switching GARCH - VSGARCH [5] (see details in [6]). All these models depend on some parameters which can be found by the maximum likelihood method. However, some parameters of the model can be fuzzy numbers [9]. Such models are called fuzzy models. In particular, there are a number of fuzzy models of autoregressive conditional heteroscedasticity. Fuzzy models are more flexible and they can be better adapted to real data. In addition, the data itself (e.g. stock market data) can be described using fuzzy numbers.

Besides, in fuzzy models not only statistical information can be taken into account, as in the classical case, but also some expert information. So one of the interesting models, in our opinion, is fuzzy asymmetric GARCH-model proposed by Hung in [8]. In this model, as in any asymmetric model, negative and positive shocks are handled differently with the help of some switching function – the characteristic function of the "almost" positive number set that "switches" the model from one parameter to another, depending on the magnitude and sign of the past shocks with respect to a threshold. In model of [8] this threshold is determined in accordance with the rules of fuzzy inference formulated by experts. As the rules in [8] the expert knowledge were considered linking NASDAQ index changes and local stock market indices changes at previous point in time with prognostic change of local indices. Examples of such rules are the following: {if the NASDAQ index falls, then the probability of a fall in the local market will increase}; {if the NASDAQ index rises, then the probability of a rise in the local market will increase} [8]. The values "index NASDAQ" or "local index" included in these rules are described with the help of linguistic variables [9], and their behavior "falls", "rises", etc. with the help of fuzzy sets that is defined by parametric membership functions. The rules of inference are fuzzy constructions of the IF-THEN type. Setting of membership functions parameters is performed by finding their values which minimize the deviation of the historical volatility of the expected historical volatility. It is interesting that this model allows us to explore the reverse problem – to establish how the macroeconomic information presented by experts is related to volatility. The Hung model was tested on the Asian stock markets, where it showed its efficiency. At the same time, the accuracy of predictions significantly differed in different markets.

This study had several purposes:

1) testing of the methodology of forecasting volatility proposed in [8] on the Russian stock market data;
2) development and research of various modifications of fuzzy asymmetric GARCH-models;
3) comparative analysis of crisp and fuzzy asymmetric GARCH-models;
4) research of the impact of some macroeconomic information on volatility.

The rest of this paper has the following structure. Section 2 describes crisp and fuzzy asymmetric GARCH-models. Section 3 contains descriptions of other fuzzy asymmetric GARCH-models introduced by the authors: a) a model in which the switching function is a s-type membership function; b) a model in which the switching function is the characteristic truth function of the comparison of the fuzzy number-histogram, constructed from the previous values of shocks, and the fuzzy threshold; c) a model in which the switching function is the index of a pair comparison of two fuzzy numbers. In these models, expert macroeconomic information is not taken into account, but statistical data is more fully taken into account. Section 4 describes a technique for determining the parameters of fuzzy asymmetric models. Section 5 shows the main results of the tests. Section 6 summarizes some conclusions from the study.

2 Crisp and fuzzy asymmetric GARCH-models

2.1 Classical GARCH-model

Let us consider a GARCH(p,q)-model defined as [2, 8]:

$$\begin{array}{l} y(t) = u(t) + c, \\ u(t) = \sqrt{\sigma(t)}\varepsilon(t), \\ \sigma^2(t) = \alpha_0 + \sum_{i=1}^{q} \alpha_i u^2(t-i) + \sum_{j=1}^{p} \beta_j \sigma^2(t-j), \end{array} \tag{1}$$

where $y(t)$ is a random variable from stock market, $\varepsilon(t)$ is a white noise process with zero mean and unit variance, σ is a conditional variance of $\varepsilon(t)$, and $\alpha_0, \alpha_i, \beta_j$, c are unknown parameters that needed to be estimated. It is assumed that:

$$\begin{array}{l} \alpha_0 > 0,\ \ \alpha_i \geq 0,\ \ i = 1, 2, \ldots, q,\ \ q > 0, \\ \beta_j \geq 0,\ \ j = 1, 2, \ldots, p,\ \ p > 0, \\ \sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1. \end{array} \tag{2}$$

As can be seen from this definition of the GARCH(p,q)-model, the current volatility depends on three components: the constant which is the product of the parameter γ (s.t. for the GARCH(1,1)-model: $\gamma + \alpha + \beta = 1$) and the long-term variance V_L, random values – "news" about volatility and past conditional volatility. However, this model assumes the same reaction to positive and negative shocks. At the same time it is proved [4] that the market reacts to positive and negative shocks in different ways. Therefore, further asymmetric GARCH-models, which differently take into account positive and negative shocks, became widespread.

2.2 Asymmetric GARCH-models

Two threshold asymmetric models were independently proposed in 1991 by Zakoyan (TGARCH) [12], and in 1993 by Glosten, Jagannathan and Runkle (GJR-GARCH) [7]. They are defined as follows:

$$\begin{array}{l} y(t) = u(t) + c, \\ u(t) = \sqrt{\sigma(t)}\varepsilon(t), \\ \sigma^d(t) = \alpha_0 + \sum_{i=1}^{q} \alpha_i^{(1)} u^d(t-i) I\{u(t-1) > 0\} + \\ + \sum_{k=1}^{q} \alpha_i^{(2)} u^d(t-k)(1 - I\{u(t-1) > 0\}) + \sum_{j=1}^{p} \beta_j \sigma^d(t-j). \end{array} \tag{3}$$

where $I(\cdot)$ is a switching function, the rest of notation is the same as in section 2.1. The difference between these two models is only in the d degree: for the TGARCH-model [12] $d = 1$, for the GJR-GARCH-model [7] $d = 2$.

Another example of an asymmetric GARCH-model is the model proposed in [5] - Volatility Switching GARCH (VSGARCH). It is specified as follows:

$$\begin{array}{l} y(t) = u(t) + [I\{u(t-1) > 0\}c^{(1)} + (1 - I\{u(t-1) > 0\})c^{(2)}], \\ u(t) = \sqrt{\sigma(t)}\varepsilon(t), \\ \sigma^2(t) = I\{u(t-1) > 0\}\left[\alpha_0^{(1)} + \sum_{i=1}^{q}\alpha_i^{(1)}u^2(t-i) + \sum_{j=1}^{p}\beta_j^{(1)}\sigma^2(t-j)\right] + \\ +(1 - I\{u(t-1) > 0\})\left[\alpha_0^{(2)} + \sum_{i=1}^{q}\alpha_i^{(2)}u^2(t-i) + \sum_{j=1}^{p}\beta_j^{(2)}\sigma^2(t-j)\right]. \end{array} \quad (4)$$

This model is similar to the models described above but its switching function-indicator has an effect on the entire formula of the model.

2.3 Fuzzy asymmetric GARCH-models

One of the most interesting, in our opinion, fuzzy asymmetric models was proposed in [8]:

$$\begin{array}{l} y(t) = u(t) + [I(t)c^{(1)} + (1 - I(t))c^{(2)}], \\ u(t) = \sqrt{\sigma(t)}\varepsilon(t), \\ \sigma^2(t) = I(t)\left[\alpha_0^{(1)} + \sum_{i=1}^{q}\alpha_i^{(1)}u^2(t-i) + \sum_{j=1}^{p}\beta_j^{(1)}\sigma^2(t-j)\right] \\ +(1 - I(t))\left[\alpha_0^{(2)} + \sum_{i=1}^{q}\alpha_i^{(2)}u^2(t-i) + \sum_{j=1}^{p}\beta_j^{(2)}\sigma^2(t-j)\right], \end{array} \quad (5)$$

where $I(t)$ is switching function such that:

$$I(t) = \begin{cases} 1, & \text{if } y(t-d) \geq r(t), \\ 0, & \text{if } y(t-d) < r(t), \end{cases} \quad (6)$$

where d – lag, $r(t)$ – threshold. This threshold value was calculated in [8] using a fuzzy model which used the tools of fuzzy numbers.

The fuzzy GARCH-model [8] consist of four components: a fuzzifier, a fuzzy rule base, a fuzzy inference engine, and a defuzzifier. It receives data (e.g. from the stock market) which are mapped into fuzzy sets. Then the fuzzy inference engine uses the fuzzy rule base to get some fuzzy output. Further, the defuzzifier maps output into the threshold value $r(t)$.

3 Modifications of an asymmetric fuzzy GARCH-model

3.1 Asymmetric GARCH-model with s-type switching function

Let us define GARCH-model with switching function of s-type as (5), but as the characteristic function we take the following:

$$I(t) = \begin{cases} 1, t > a + \Delta, \\ \frac{t-(a-\Delta)}{2\Delta}, a - \Delta \leq t \leq a + \Delta, \\ 0, t < a - \Delta, \end{cases} \quad (7)$$

where a, Δ, are parameters that are estimated with the model coefficients by the maximum likelihood estimation (MLE).

The difference between this model and Hung's model [8] is that here the characteristic function can take all values from 0 to 1. It allows us to give some weight for positive and negative shocks.

3.2 Asymmetric GARCH-model with the characteristic function of comparing the fuzzy number-histogram and the fuzzy threshold

Characteristic function for this model:

$$I(t) = \begin{cases} 1, r^d(t) \succ h, \\ 0, \text{otherwise}, \end{cases} \quad (8)$$

where $r^d(t)$ is the fuzzy number-histogram constructed on last d values of $y(t)$, h is the fuzzy threshold, $\succ$ is some operation of comparing fuzzy numbers [10].

In this study, as the fuzzy threshold we used a symmetrical triangular fuzzy number [10] which is defined by two parameters: mean value and amplitude. These parameters are found with the model coefficients. A comparison of the fuzzy threshold and the fuzzy number-histogram was made using the mean value of a fuzzy number support.

3.3 Asymmetric GARCH-model with a switching function of the index of fuzzy numbers pairwise comparison

In this model, as a switching function we use some index of fuzzy numbers pairwise comparison R [10]:

$$I(t) = R(r^d(t), h), \quad (9)$$

where $r^d(t)$ and h are the fuzzy number-histogram and the fuzzy threshold defined in the previous section. As an index of pairwise comparison we used the popular Baas-Kwakernaak index [1]:

$$R(r^d(t), h) = \sup_{i \geq j} min\{\mu_{r^d(t)}(i), \mu_h(j)\}, \quad (10)$$

where $\mu_{r^d(t)}$ and μ_h are membership functions of the fuzzy number-histogram and the fuzzy threshold, respectively.

4 Asymmetric model parameters estimation

In this study, the volatility of MICEX and RTS indices was investigated. Rules for forecasting were determined separately for each index. Formally, the rules were the same but their coefficients in a fuzzy system were different. These coefficients were calculated using MLE. It is worth noting that the sum of coefficients of rules is not always equal to 1. Rules and their coefficients are shown in Tab.1 and Tab.2.

Table 1. Rules and their coefficients for MICEX index.

	MICEX index fall	MICEX index neutral	MICEX index rise
Dollar fall against ruble	0	0.3	0.4
Dollar neutral against ruble	0	0.2	0.3
Dollar rise against ruble	0	0.3	0.4
MICEX index fall	0	0.3	0.4
MICEX index neutral	0	0.2	0.3
MICEX index rise	0	0.4	0.7

Table 2. Rules and their coefficients for RTS index.

	RTS index fall	RTS index neutral	RTS index rise
Dollar fall against ruble	0	0	0.1
Dollar neutral against ruble	0	0.1	1
Dollar rise against ruble	0	0	0.1
RTS index fall	0	0	0.1
RTS index neutral	1	0.6	0.3
RTS index rise	0	0	0.1

In [8] were also estimated membership functions of input (from stock market) and output (threshold) variables. However, we used once defined membership functions. They were set so that their "tails" intersected in some neighbourhood of zero.

In addition, new models have been tested which are modeifications of GJR-GARCH-model [7]. Their essence is that the models formula is set in the same way as in the GJR-GARCH-model but the models use switching functions of other models: the function from [8], the s-type function, the fuzzy-histogram and fuzzy threshold comparison function, the function of the index of fuzzy numbers pairwise comparison. Tab.3 gives estimates of the fuzzy asymmetric Hung GJR-GARCH-model for MICEX and RTS indices.

Table 3. Estimated coefficients of Hung GJR-GARCH-model.

	MICEX index	RTS index
c	8.46E-4	5.60E-4
a_0	4.83E-6	4.81E-6
$a_1^{(1)}$	2.07E-6	6.21E-7
$a_1^{(2)}$	0.061	0.045
b	0.919	0.955

5 Testing results

The models were tested on the MICEX and RTS indices, the dollar exchange rate for the period from January 1, 2015 to December 31, 2016. The dates for the indices and the dollar exchange rate were matched where there were gaps for any instrument. The models were trained on 95% of the data and the prediction was carried out for the next 5 days.

Three types of errors were used to compare the models: mean square forecast error (MSFE), mean absolute forecast error (MAFE) and largest absolute forecast error (LAFE) during a prediction period. Furthermore, a forecast error to volatility ratio was considered in order to understand how predicted and historical volatility differ.

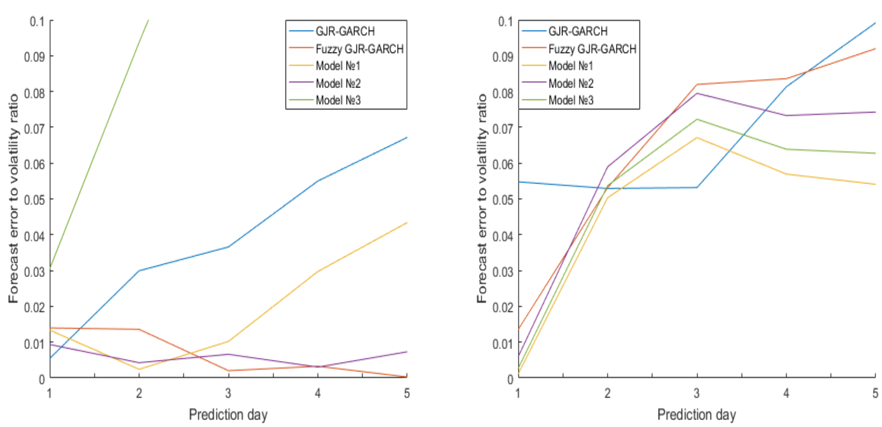

Fig. 1. Ratios of prediction errors and historical volatility for MICEX(left) and RTS(right) index.

These ratios are in Fig.1, where "Model 1" is asymmetric GARCH-model withs-type switching function, "Model 2" is asymmetric GARCH-model with the characteristic function of comparing the fuzzy number-histogram and the fuzzy threshold, and "Model 3" is Asymmetric GARCH-model with a switching function of the index of fuzzy numbers pairwise comparison.

These results show that the error percentage is very small. Moreover, the proposed models 1 and 2 show results comparable to the fuzzy GJR-GARCH-model and in most days better than the GJR-GARCH-model for RTS index and MICEX index, respectively.

Tab.4 and Tab.5 present the errors of these models. It can be concluded that models 1 and 2 (in the form of the GJR-GARCH-model) proved to be the best on RTS and MICEX indices, respectively. The fuzzy model also showed good results, but, as noted above, membership functions were not optimized, and the rules were obtained not as information from experts but by learning the model.

Table 4. Models errors on MICEX index.

	MSFE	MAFE	LAFE
GJR-GARCH	1.25E-7	3.10E-4	5.37E-4
Fuzzy GJR-GARCH	4.99E-9	5.27E-5	1.11E-4
Model 1	3.90E-8	1.58E-4	3.47E-4
Model 2	2.69E-9	4.87E-5	7.42E-5
Model 3	1.95E-6	1.22E-3	2.14E-3

Table 5. Models errors on MICEX index.

	MSFE	MAFE	LAFE
GJR-GARCH	7.20E-7	8.13E-4	1.21E-3
Fuzzy GJR-GARCH	7.35E-7	7.80E-4	1.12E-3
Model 1	3.84E-7	5.53E-4	8.13E-4
Model 2	6.03E-7	7.03E-4	9.63E-4
Model 3	4.70E-7	6.15E-4	8.75E-4

6 Conclusion

In this paper, a comparative analysis of crisp and fuzzy asymmetric GARCH-models is made with respect to forecasting the volatility of Russian stock indices. Various variants of construction of a switching function using the theory of fuzzy sets are considered. In some fuzzy models, the switching function is constructed taking into account the aggregation of expert macroeconomic information. Instead of real information from experts ”pseudo-expert” information obtained as a result of training the system on historical data was used in the work. The results of testing models on MICEX and RTS indices showed:

1) practically all considered fuzzy asymmetric GARCH-models have better prognostic ability than their crisp analogues;
2) using of expert information does not significantly improve the result;
3) the predictive ability of various fuzzy models is significantly different on MICEX and RTS indices.

In terms of further research, it is interesting as optimizing fuzzy models for all parameters, so more complete accounting of information from various sources (and not only expert).

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