The Qualitative Characteristics of Combining Evidence with Discounting *

Alexander Lepskiy

Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow, 101000, Russia

alex.lepskiy@gmail.com

September 2016

Abstract

The qualitative characteristics of the combining evidence with the help of Dempster's rule with discounting is studied in this paper in the framework of Dempster-Shafer theory. The discount coefficient (discounting rate) characterizes the reliability of information source. The conflict between evidence and change of ignorance after applying combining rule are considered in this paper as important characteristics of quality of combining. The quantity of ignorance is estimated with the help of linear imprecision index. The set of crisp and fuzzy discounting rates for which the value of ignorance after combining does not increases is described.

Keywords: belief functions, discount method, imprecise index

1 Introduction

The study of combining rules of evidence occupies an important place in the belief functions theory. A combining rule puts in correspondence to two or more evidences the one evidence. Dempster's rule [5] was the first from combining rules. The review of some popular combining rules can be found in [11]. There is no combining rule which give a plausible aggregation of information in all cases regardless of context. The prognostic quality of combining evidence is evaluated with the help of some characteristics. The reliability of sources of information, the conflict measure of evidence [8], the degree of independence of evidence are a priori characteristics of quality of combining. The amount of change of ignorance after the use of a combining rule is the most important a posteriori characteristic [9]. The amount of ignorance contained in evidence may be

*The final publication is available at
https://link.springer.com/chapter/10.1007/978-3-319-42972-4_39
Lepskiy A. (2017) The Qualitative Characteristics of Combining Evidence with Discounting. In: Ferraro M. et al. (eds) Soft Methods for Data Science. Advances in Intelligent Systems and Computing, vol 456. Springer, Cham, pp.311-318.

estimated with the help of imprecision indices [2]. The generalized Hartley's measure is an example of such index [6]. It is known, for example, that the amount of ignorance does not increase when used Dempster's rule for non-conflicting evidences. Dempster's rule can be considered as an optimistic rule in this sense [9]. On the contrary, Dubois and Prade's disjunctive consensus rule [7] has a pessimistic character in the sense that amount of ignorance does not decrease after applying such a rule.

The discount method is one of the approaches where the reliability of information source is taken into account. This method was proposed by Shafer in [12]. The discount coefficient (discounting rate) characterizes the reliability of information source. The discount method with Dempster's rule may be pessimistic rule or optimistic rule in depending on the values of discounting rates. The generalizations of the discount method were considered in several papers. In particular, Smets in [13] introduced a family of combination rules known as α-junctions. Pichon and Denoeux [10] have established the link between the parameter of α-junction and reliability of information sources.

In this paper we will find conditions on the discount rates for which the amount of ignorance after applying Dempster's rule is not increased, i.e. this rule will be still optimistic in spite of unreliable information sources. This problem is solved in general case of conflicting evidences and crisp discounting rates as well as in the case of non-conflicting evidences and fuzzy discounting rates. In addition, the problem of finding such discount rates for which a conflict of evidence will not be greater than a certain threshold and the quality of ignorance after the combination will not increase is formulated and solved.

2 Belief Function Basics

Let X be a finite universal set and 2^X is a set of all subsets of X. We consider the belief function [12] $g : 2^X \to [0,1]$. The value $g(A)$, $A \in 2^X$, is interpreted as a degree of confidence that the true alternative of X belongs to set A. A belief function g is defined with the help of so called mass function $m_g : 2^X \to [0,1]$ that satisfy the conditions [12]: $m_g(\emptyset) = 0$, $\sum_{A \subseteq X} m_g(A) = 1$. Then $g(A) = \sum_{B:\, B \subseteq A} m_g(B)$. Let the set of all belief functions on 2^X be denoted by $Bel(X)$.

Conversely the mass function m_g can be calculated by using the belief function g with the help of so-called Möbius transform of g: $m_g(B) = \sum_{A: A \subseteq B} (-1)^{|B \setminus A|} g(A)$. The belief function $g \in Bel(X)$ may be represented with the help of so called categorical belief functions $\eta_{\langle B \rangle}(A) = \begin{cases} 1, & B \subseteq A, \\ 0, & B \not\subseteq A, \end{cases}$ $A \subseteq X$, $B \neq \emptyset$. Then $g = \sum_{B \in 2^X \setminus \{\emptyset\}} m_g(B) \eta_{\langle B \rangle}$. The set $\{\eta_{\langle B \rangle}\}$, $B \in 2^X \setminus \{\emptyset\}$ is a basis in set $Bel(X)$ in the sense that any belief function $g \in Bel(X)$ is represented by uniquely as a convex combination of primitive measures $\{\eta_{\langle B \rangle}\}_{B \in 2^X \setminus \{\emptyset\}}$. The subset $A \in 2^X$ is called a focal element if $m(A) > 0$. Let $\mathcal{A}$ be a set of all focal elements. The pair $F = (\mathcal{A}, m)$ is called a body of evidence. We will denote through $\mathcal{A}(g)$ and $F(g)$ the set of all focal elements and the body of evidence correspondingly related with the belief function g. Let us have two bodies of evidence $F(g_1) = (\mathcal{A}(g_1), m_{g_1})$ and $F(g_2) = (\mathcal{A}(g_2), m_{g_2})$ which related with the belief functions $g_1, g_2 \in Bel(X)$. For example, it can be evidences which were received from two information sources. Then the task of combining of these two evidence in one evidence with the help of some operator $\varphi : Bel^2(X) \to Bel(X)$, $g = \varphi(g_1, g_2)$, is an actual problem.

Dempster's rule was the first from combining rules. This rule was introduced in [5] and generalized in [12] for combining arbitrary independent evidence. This rule is defined as $g = \varphi_D(g_1, g_2) = \sum_{A\in 2^X\setminus\{\emptyset\}} m_g(A)\eta_{\langle A\rangle}$, where

$$m_g(A) = \frac{1}{1-K}\sum_{B\cap C=A} m_{g_1}(B)m_{g_2}(C),\quad A \neq \emptyset,\quad m_g(\emptyset) = 0, \qquad (1)$$

$$K = K(g_1, g_2) = \sum_{B\cap C=\emptyset} m_{g_1}(B)m_{g_2}(C).$$

The value $K(g_1, g_2)$ characterizes the amount of conflict in two information sources which determined with the help of bodies of evidence $F(g_1)$ and $F(g_2)$. If $K(g_1, g_2) = 1$ then it means that information sources are absolutely conflict and Dempster's rule cannot be applied. The discounting of mass function was introduced by Shafer [12] for accounting of reliability of information. The main idea consists in the use of coefficient $\alpha \in [0, 1]$ for discounting of mass function:

$$m^\alpha(A) = (1-\alpha)m(A),\quad A \neq X,\quad m^\alpha(X) = \alpha + (1-\alpha)m(X). \qquad (2)$$

The coefficient α is called the discounting rate. The discounting rate characterized the degree of reliability of information. If $\alpha = 0$ then it means that information source is absolutely reliable. If $\alpha = 1$ then it means that information source is absolutely non-reliable. Dempster's rule (1) applies after discounting of mass functions of two evidences in general with different discounting rates.

The following Dubois and Prade's disjunctive consensus rule is a dual to Dempster's rule [7]: $g = \varphi_{DP}(g_1, g_2) = \sum_{A\in 2^X\setminus\{\emptyset\}} m_g(A)\eta_{\langle A\rangle}$, where $m_g(A) = \sum_{B\cup C=A} m_{g_1}(B)m_{g_2}(C)$, $A \in 2^X$.

3 Estimation of ignorance associated with the belief function

Let us have source of information and this information is described by a belief function $g \in Bel(X)$. The belief function g defines the information with some degree of uncertainty. There are few approaches to definition of uncertainty measure in the evidence theory. We will follow the approach described in work [2]. This approach based on the notion of imprecision index.

Let us know only that true alternative belong to the non empty set $B \subseteq X$. This situation may be described with the help of categorical belief function $\eta_{\langle B\rangle}(A)$, $A \subseteq X$, which gives the lower probability of an event $x \in A$. The degree of uncertainty of such function is described by the well-known Hartley measure $H(\eta_{\langle B\rangle}) = \log_2|B|$, which characterized the degree of information uncertainty about belonging of true alternative to set $B \subseteq X$.

The following construction is a generalization of above situation. Let $g = \sum_{B\in 2^X} m_g(B)\eta_{\langle B\rangle} \in Bel(X)$. Then the generalized Hartley measure [6] from g is defined as $GH(g) = \sum_{B\in 2^X\setminus\{\emptyset\}} m_g(B)\log_2|B|$. The generalized Hartley measure is an example of the following general notion.

Definition 3.1. [2]. A functional $f : Bel(X) \to [0, 1]$ is called imprecision index if the following conditions are fulfilled: 1) if g be a probability measure then $f(g) = 0$; 2) $f(g_1) \geq f(g_2)$ for all $g_1, g_2 \in Bel(X)$ such that $g_1 \leq g_2$ (i.e. $g_1(A) \leq g_2(A)$ for all $A \in 2^X$); 3) $f\left(\eta_{\langle X\rangle}\right) = 1$.

An imprecision index f on $Bel(X)$ is called linear if for any linear combination $\sum_{j=1}^k \alpha_j g_j \in Bel(X)$, $\alpha_j \in \mathbb{R}$, $g_j \in Bel(X)$, $j = 1, ..., k$, we have $f\left(\sum_{j=1}^k \alpha_j g_j\right) = \sum_{j=1}^k \alpha_j f(g_j)$.

Since any linear functional f on $Bel(X)$ is defined uniquely by its values on a set of primitive measures $\{\eta_{\langle B\rangle}\}_{B\in 2^X\setminus\{\emptyset\}}$, then it allows us to define f with the help of set function $\mu_f : 2^X \to \mathbb{R}$ by the rule $\mu_f(B) = f\left(\eta_{\langle B\rangle}\right)$, $B \in 2^X\setminus\{\emptyset\}$. It is easy to see that monotonicity of set function $\mu_f(B)$ follows from antimonotonicity of functional f (condition 2) of Definition 3.1) and inequality $\eta_{\langle B'\rangle} \geq \eta_{\langle B''\rangle}$ if $B' \subseteq B''$. We set by definition that $\mu_f(\emptyset) = 0$ for every imprecision index f.

The different representations of imprecision index were found in [2]. In this paper we will use the following representation.

Proposition 3.2. *A functional $f : Bel(X) \to [0, 1]$ is a linear imprecision index on $Bel(X)$ iff $f(g) = \sum_{B\in 2^X\setminus\{\emptyset\}} m_g(B)\mu_f(B)$, where set function μ_f satisfies the conditions: 1) $\mu_f(\{x\}) = 0$ for all $x \in X$; 2) $\mu_f(X) = 1$; 3) $\sum_{B: A\subseteq B} (-1)^{|B\setminus A|}\mu_f(B) \leq 0$ for all $A \neq \emptyset, X$.*

Example 1. Let $\mu_f(B) = \psi(|B|)$. Then functions $\psi(t) = \ln t/\ln|X|$, $\psi(t) = (t-1)^s/(|X|-1)^s$, $s \in (0, 1]$ satisfy the all conditions of Proposition 3.2.

4 Change of ignorance after combining with the crisp discount rates

Assume that we have two information sources which are defined by the bodies of evidence $F(g_1) = (\mathcal{A}(g_1), m_{g_1})$ and $F(g_2) = (\mathcal{A}(g_2), m_{g_2})$ correspondingly and which related with the belief functions $g_1, g_2 \in Bel(X)$. If we apply some combining rule φ to the pair of belief functions $g_1, g_2 \in Bel(X)$ then we get a new belief function $g = \varphi(g_1, g_2)$. We have a question about changing of the amount of ignorance after applying combining rule φ. We will estimate the quantity of ignorance with the help of imprecision index f.

Definition 4.1. A combining rule φ is called optimistic (pessimistic) rule with respect to imprecision index f, if $f(g) \leq \min\limits_{i\in 1,2} f(g_i)$ $\left(f(g) \geq \max\limits_{i\in 1,2} f(g_i)\right)$ for all $g_1, g_2 \in Bel(X)$.

In other words, the optimistic rule does not increase the amount of ignorance, but the pessimistic rule does not decrease the amount of ignorance. It is known [7], [9] that Dempster's rule is an optimistic rule with respect to any linear imprecision index, but Dubois and Prade's disjunctive consensus rule is a pessimistic rule.

Proposition 4.2. *Let f be a linear imprecision index on $Bel(X)$. Then: 1) $f(g) \leq \min_i f(g_i)$, if $g = \varphi_D(g_1, g_2)$ and evidences related with the belief functions $g_1, g_2 \in Bel(X)$ are non-conflicting; 2) $f(g) \geq \max_i f(g_i)$, if $g = \varphi_{DP}(g_1, g_2)$.*

Now we investigate on pessimism-optimism Dempster's rule with discounting. Let

$$g_1 = \sum\nolimits_{A \in 2^X \setminus \{\emptyset\}} m_{g_1}(A)\eta_{\langle A\rangle}, \quad g_2 = \sum\nolimits_{A \in 2^X \setminus \{\emptyset\}} m_{g_2}(A)\eta_{\langle A\rangle}.$$

Each of two information sources has its own reliability (discount rate) $\alpha, \beta \in [0,1]$ correspondingly in the sense of discounting method (2). We obtain two new belief functions taking into account discount rates:

$$g_1^{(\alpha)} = \sum\nolimits_{A \in 2^X \setminus \{\emptyset\}} m_{g_1}^{(\alpha)}(A)\eta_{\langle A\rangle}, \quad g_2^{(\beta)} = \sum\nolimits_{B \in 2^X \setminus \{\emptyset\}} m_{g_2}^{(\beta)}(B)\eta_{\langle B\rangle},$$

where $m_{g_1}^{(\alpha)}(A) = (1-\alpha)m_{g_1}(A)$, $A \neq X$, $m_{g_1}^{(\alpha)}(X) = \alpha + (1-\alpha)m_{g_1}(X)$ and $m_{g_2}^{(\beta)}$ calculated similarly. We note that

$$g_1^{(\alpha)} = \sum\nolimits_{A \in 2^X \setminus \{\emptyset\}} m_{g_1}(A)\eta_{\langle A\rangle}^{(\alpha)}, \quad g_2^{(\beta)} = \sum\nolimits_{B \in 2^X \setminus \{\emptyset\}} m_{g_2}(B)\eta_{\langle B\rangle}^{(\beta)}, \quad (3)$$

where $\eta_{\langle A\rangle}^{(\alpha)} = (1-\alpha)\eta_{\langle A\rangle} + \alpha\eta_{\langle X\rangle}$ and $\eta_{\langle B\rangle}^{(\beta)}$ calculated similarly. We assume that evidences $F(g_1)$ and $F(g_2)$ are non-conflicting, i.e. $K = K(g_1, g_2) = 0$. Then $K\left(g_1^{(\alpha)}, g_2^{(\beta)}\right) = 0$. If we apply Dempster's rule φ_D to the pair $g_1^{(\alpha)}, g_2^{(\beta)}$ of belief functions then we get a new belief function $g_{\alpha,\beta} = \varphi_D(g_1^{(\alpha)}, g_2^{(\beta)})$. Dempster's rule $\varphi_D(g_1^{(\alpha)}, g_2^{(\beta)})$ is a linear rule for every argument for non-conflicting evidences. Therefore we get from representations (3)

$$\varphi_D(g_1^{(\alpha)}, g_2^{(\beta)}) = \sum_{A \in \mathcal{A}(g_1)} \sum_{B \in \mathcal{A}(g_2)} m_{g_1}(A) m_{g_2}(B) \varphi_D\left(\eta_{\langle A\rangle}^{(\alpha)}, \eta_{\langle B\rangle}^{(\beta)}\right). \quad (4)$$

We have $A \cap B \neq \emptyset$ for every pair $A \in \mathcal{A}(g_1)$, $B \in \mathcal{A}(g_2)$ in case of non-conflicting evidences. Consequently we get

$$\varphi_D\left(\eta_{\langle A\rangle}^{(\alpha)}, \eta_{\langle B\rangle}^{(\beta)}\right) = (1-\alpha)(1-\beta)\eta_{\langle A \cap B\rangle} + (1-\alpha)\beta\eta_{\langle A\rangle} + \alpha(1-\beta)\eta_{\langle B\rangle} + \alpha\beta\eta_{\langle X\rangle}.$$

Consequently we have from (4),

$$g_{\alpha,\beta} = \varphi_D(g_1^{(\alpha)}, g_2^{(\beta)}) = \sum_{A \in \mathcal{A}(g_1)} \sum_{B \in \mathcal{A}(g_2)} m_{g_1}(A) m_{g_2}(B) \varphi_D\left(\eta_{\langle A\rangle}^{(\alpha)}, \eta_{\langle B\rangle}^{(\beta)}\right) =$$

$$(1-\alpha)(1-\beta) \sum_{A \in \mathcal{A}(g_1)} \sum_{B \in \mathcal{A}(g_2)} m_{g_1}(A) m_{g_2}(B) \eta_{\langle A \cap B\rangle} + (1-\alpha)\beta \sum_{A \in \mathcal{A}(g_1)} m_{g_1}(A)\eta_{\langle A\rangle} +$$

$$(1-\beta)\alpha \sum_{B\in\mathcal{A}(g_2)} m_{g_2}(B)\eta_{\langle B\rangle} + \alpha\beta\eta_{\langle X\rangle}.$$

Therefore, a new belief function $g_{\alpha,\beta}$ has the following expression through initial functions $g_1, g_2 \in Bel(X)$ and the belief function $g = \varphi_D(g_1, g_2)$ obtained without discounting

$$g_{\alpha,\beta} = \varphi_D(g_1^{(\alpha)}, g_2^{(\beta)}) = (1-\alpha)(1-\beta)g + (1-\alpha)\beta g_1 + \\ + (1-\beta)\alpha g_2 + \alpha\beta\eta_{\langle X\rangle}. \quad (5)$$

We have a question about changing of the amount of ignorance after applying Dempster's rule with discounting. We will estimate the quantity of ignorance with the help of linear imprecision index f. Dempster's rule is an optimistic rule (i.e. $f(g) \le \min_i f(g_i),$) for non-conflicting and reliable information sources ($\alpha, \beta = 0$) with respect to any linear imprecision index as it follows from Proposition 4.2.

If we use non-reliable information sources ($\alpha, \beta \ne 0$) then imprecision index $f(g_{\alpha,\beta})$ of new belief function $g_{\alpha,\beta}$ could be greater than imprecision indices of initial functions $f(g_i)$, $i = 1, 2$. We will find the conditions on discounting rates for which the amount of ignorance will not increase after applying Dempster's rule with discounting. We obtain from (5) with account of linearity of index f and normalization condition $f(\eta_{\langle X\rangle}) = 1$ that

$$f(g_{\alpha,\beta}) = (1-\alpha)(1-\beta)f(g) + (1-\alpha)\beta f(g_1) + (1-\beta)\alpha f(g_2) + \alpha\beta. \quad (6)$$

The function $f(g_{\alpha,\beta})$ can be rewritten in the form

$$f(g_{\alpha,\beta}) = f(g) + \alpha\Delta_2 + \beta\Delta_1 + \alpha\beta(\Delta - \Delta_1 - \Delta_2), \quad (7)$$

where $\Delta_i = f(g_i) - f(g)$, $i = 1, 2$ is a changing of ignorance of i-th information source after applying Dempster's rule (without of discounting), $\Delta = 1 - f(g)$. Note that we have $\Delta_i \ge 0$, $i = 1, 2$ in any non-conflicting case and we have $\Delta \ge \Delta_i$, $i = 1, 2$ in any case. Then the condition $f(g_{\alpha,\beta}) \le f(g_i)$, $i = 1, 2$ is equivalent to inequality

$$\alpha\Delta_2 + \beta\Delta_1 + \alpha\beta(\Delta - \Delta_1 - \Delta_2) \le \min\{\Delta_1, \Delta_2\}. \quad (8)$$

Let $Ign_0 = Ign_0(g_1, g_2)$ be a set of all pair $(\alpha, \beta) \in [0,1]^2$ which satisfy inequality (8) for given belief functions $g_1, g_2 \in Bel(X)$. Note that the set $Ign_0(g_1, g_2)$ is a star domain (or star-convex set, star-shaped or radially convex set) [4] with star center in the origin, i.e. if $(\alpha_0, \beta_0) \in Ign_0(g_1, g_2)$, then $(t\alpha_0, t\beta_0) \in Ign_0(g_1, g_2)$ for all $t \in [0, 1]$.

Indeed, let $(\alpha_0, \beta_0) \in Ign_0(g_1, g_2)$. We will show that the point $(t\alpha_0, t\beta_0) \in Ign_0(g_1, g_2)$ for all $t \in [0, 1]$. We denote the function of the left side of the inequality (8) in point $(t\alpha_0, t\beta_0)$through $\psi(t)$. Then it is sufficient to show that the function $\psi(t)$ does not decrease on segment $[0, 1]$. We have

$$\psi'(t) = \alpha_0\Delta_2 + \beta_0\Delta_1 + 2t\alpha_0\beta_0(\Delta - \Delta_1 - \Delta_2).$$

If $\Delta - \Delta_1 - \Delta_2 \ge 0$, then $\psi'(t) \ge \alpha_0\Delta_2 + \beta_0\Delta_1 \ge 0$ for all $t \in [0, 1]$. If $\Delta - \Delta_1 - \Delta_2 < 0$, then we have for $t \in [0, 1]$

$$\psi'(t) \ge \alpha_0\Delta_2 + \beta_0\Delta_1 - 2\alpha_0\beta_0(\Delta_1 + \Delta_2 - \Delta) \ge$$

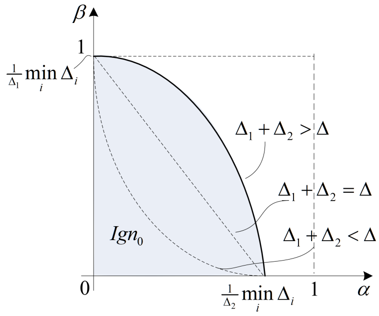

Figure 1: The general view of the set $Ign_0 = Ign_0(g_1, g_2)$ under combining non-conflict evidences (the case $\Delta_1 < \Delta_2$ is shown).

$$2\sqrt{\alpha_0\beta_0\Delta_1\Delta_2} - 2\alpha_0\beta_0(\Delta_1 + \Delta_2 - \Delta) \geq$$

$$2\sqrt{\alpha_0\beta_0}\left(\sqrt{\Delta_1\Delta_2} + \Delta - \Delta_1 - \Delta_2\right) \geq 0,$$

because, for example, the inequality is true in the case $\Delta_1 \geq \Delta_2$ $\sqrt{\Delta_1\Delta_2}+ \Delta - \Delta_1 - \Delta_2 = \sqrt{\Delta_2}\left(\sqrt{\Delta_1} - \sqrt{\Delta_2}\right) + (\Delta - \Delta_1) \geq 0$ due to $\Delta \geq \Delta_i$, $i = 1, 2$.

The general view of the set $Ign_0(g_1, g_2)$ is shown in Figure 1.

We have the following result in the general case of conflicting evidence (i.e. $K = K(g_1, g_2) \neq 0$) . We consider further the case of conflicting evidence (i.e. $K = K(g_1, g_2) \neq 0$). In this case we obtain the following value of conflict after discounting

$$K_{\alpha,\beta} = K\left(g_1^{(\alpha)}, g_2^{(\beta)}\right) = (1-\alpha)(1-\beta)\sum_{A\cap B=\emptyset} m_{g_1}(A)m_{g_2}(B)=$$

$$= (1-\alpha)(1-\beta)K. \tag{9}$$

Then

$$f(g_{\alpha,\beta})= \frac{(1-\alpha)(1-\beta)(1-K)f(g) + (1-\alpha)\beta f(g_1) + (1-\beta)\alpha f(g_2) + \alpha\beta}{1-(1-\alpha)(1-\beta)K} =$$

$$= f(g) + \frac{\alpha\Delta_2 + \beta\Delta_1 + \alpha\beta(\Delta - \Delta_1 - \Delta_2)}{1-(1-\alpha)(1-\beta)K}.$$

The same equality can be obtained directly from (6) (but not strictly) taking into account a linearity of the functional f. Thus, the following statement is true.

Proposition 4.3. *Dempster's rule with discounting $(\alpha, \beta) \in [0, 1]^2$ is optimistic rule with respect to linear imprecision index f (i.e. $f(g_{\alpha,\beta}) \leq \min_i f(g_i)$) iff*

$$\alpha\Delta_2+\beta\Delta_1+\alpha\beta(\Delta-\Delta_1-\Delta_2) \leq (1-(1-\alpha)(1-\beta)K)\min\{\Delta_1, \Delta_2\}. \tag{10}$$

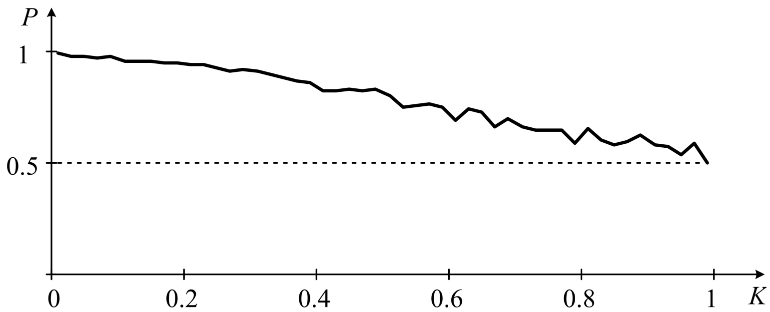

Figure 2: A graph of statistical estimates of the probability $P = \tilde{P}\{\min\{\Delta_1, \Delta_2\} \geq 0|\, K\}$ from conflict K.

Let $Ign_K = Ign_K(g_1, g_2)$ be a set of all pair $(\alpha, \beta) \in [0,1]^2$, which satisfy inequality (10) for given belief functions $g_1, g_2 \in Bel(X)$, which have conflict $K = K(g_1, g_2)$. It is easy to see from (10) that $Ign_{K'} \subseteq Ign_{K''} \subseteq Ign_0$, if $K' \geq K''$ under condition $\Delta_i = f(g_i) - f(g) \geq 0$, $i = 1, 2$.

One can show that the set $Ign_K = Ign_K(g_1, g_2)$ is a star domain with star center in the origin in case $\Delta_i \geq 0$, $i = 1, 2$, iff $\Delta - \Delta_1 - \Delta_2 \geq 0$.

Remark 1. The statistical analysis shows that probability $P\{\min\{\Delta_1, \Delta_2\} \geq 0|K\}$ is decreased with growth of K but it will be greater than 0,5 in case of uniform independent distribution of evidence (mass function and focal sets). A graph of statistical estimates of the probability $P = \tilde{P}\{\min\{\Delta_1, \Delta_2\} \geq 0|\, K\}$ from conflict K in the case $|X| = 7$, $|\mathcal{A}(g_1)| = |\mathcal{A}(g_2)| = 2$, is shown on Figure 2 for uniform independent generation of focal sets and mass function.

Moreover we have that $\min\{\Delta_1, \Delta_2\} > 0$ approximately in 95% of cases, but $\max\{\Delta_1, \Delta_2\} < 0$ approximately in 0,1% of cases under the same conditions of statistical tests.

The value of conflict is changed if the evidence are discounted in according to the formula (9). The value of conflict after discounting is equal $K_{\alpha,\beta} = K\left(g_1^{(\alpha)}, g_2^{(\beta)}\right) = (1-\alpha)(1-\beta)K$. If the discount rates are increased then the value of conflict between the evidence is decreased. The problem of description of all pair $(\alpha, \beta) \in [0,1]^2$ for given belief functions $g_1, g_2 \in Bel(X)$ for which the conflict $K_{\alpha,\beta}$ is not greater some threshold value $K_{\max} \leq K$ (i.e. $K_{\alpha,\beta} = (1-\alpha)(1-\beta)K \leq K_{\max}$) can be formulated. We denote this set through $Confl_K(K_{\max})$.

The problem of description of reliability of information sources (discounting rates) for which the aggregation with the help of Dempster's rule will not lead to an increase of ignorance ($(\alpha, \beta) \in Ign_K$) but a conflict will not be great ($(\alpha, \beta) \in Confl_K(K_{\max})$) is an actual problem. This set is defined as $Ign_K \cap Confl_K(K_{\max})$.

The example of set $Ign_K \cap Confl_K(K_{\max})$ is shown on Figure 3.

Now the problem of finding of points-reliabilities $(\alpha, \beta) \in Ign_K \cap Confl_K(K_{\max})$ for which the imprecision index $f(g_{\alpha,\beta})$ after combining

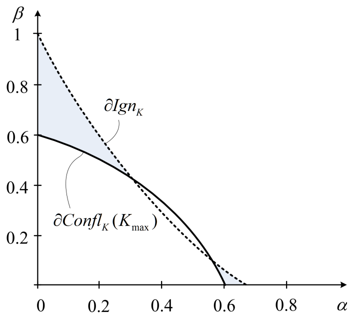

Figure 3: The example of set $Ign_K \cap Confl_K(K_{\max})$

will be minimal can be formulated:

$$f(g_{\alpha,\beta}) \to \min, \quad (\alpha, \beta) \in Ign_K \cap Confl_K(K_{\max}). \tag{11}$$

This problem is an actual if we have several pairs of conflicting information sources with different reliabilities. We must choose the best pair for combining. Note that the formulation of the problem (11) can be considered as an optimization problem of finding of combining rule from parametric family of rules $\{g_{\alpha,\beta}\}_{\alpha,\beta\in[0,1]}$, for which the ignorance will be minimal under the condition that the conflict is not greater some threshold value $K_{\max}$. The generalized statement of the problem is considered in [3].

5 Change of ignorance after combining with fuzzy discount rates

Assume that reliabilities of information sources α and β are not known precisely but we have a fuzzy numbers $\tilde{\alpha}$ and $\tilde{\beta}$. Then the imprecision index $f(g_{\tilde{\alpha},\tilde{\beta}})$ will be by a fuzzy number also and, for example, in case of non-conflicting evidence (see (7)) $f(g_{\tilde{\alpha},\tilde{\beta}})$ is equal

$$f(g_{\tilde{\alpha},\tilde{\beta}}) = f(g) + \tilde{\alpha}\Delta_2 + \tilde{\beta}\Delta_1 + \tilde{\alpha}\tilde{\beta}(\Delta - \Delta_1 - \Delta_2).$$

Then we can formulate the problem of finding of the fuzzy numbers $\tilde{\alpha}$ and $\tilde{\beta}$ for which $f(g_{\tilde{\alpha},\tilde{\beta}}) \leq_I f(g_i)$, $i = 1, 2$, where $\leq_I$ is a some relation of comparison of fuzzy numbers [14].

Example. Let $\tilde{\alpha}$ and $\tilde{\beta}$ are by triangular fuzzy numbers of the form $\tilde{\alpha} = (\alpha - \delta, \alpha, \alpha + \delta)$ and $\tilde{\beta} = (\beta - \omega, \beta, \beta + \omega)$ correspondingly. We will use the method Adamo [1] for comparison of the fuzzy numbers $\tilde{u}$ and $\tilde{v}$. Let $\tilde{u}_\gamma = \{t | \mu_{\tilde{u}}(t) \geq \gamma\}$ be a γ-cut of fuzzy number $\tilde{u}$ with relationship function $\mu_{\tilde{u}}$ and $\tilde{u}_\gamma = [l_{\tilde{u}}(\gamma), r_{\tilde{u}}(\gamma)]$. The fuzzy number $\tilde{u}$ does not exceed the fuzzy number $\tilde{v}$ with respect to the method Adamo ($\tilde{u} \leq_A \tilde{v}$), if $r_{\tilde{u}}(\gamma) \leq r_{\tilde{v}}(\gamma)$ for given (fixed) level $\gamma \in (0, 1]$. The level γ characterizes a measure of risk of the wrong decision. Then

$$f(g_{\tilde{\alpha},\tilde{\beta}}) \leq_I f(g_i) \Leftrightarrow r_{f(g_{\tilde{\alpha},\tilde{\beta}})}(\gamma) \leq \min_i f(g_i),$$

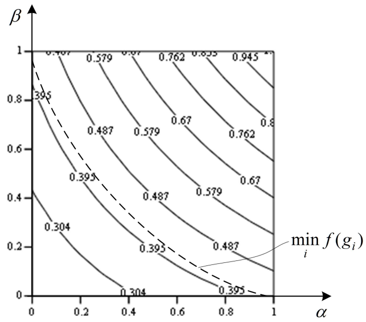

Figure 4: A contour map of function $r_{f(g_{\tilde{\alpha},\tilde{\beta}})}(\gamma)$.

where $r_{f(g_{\tilde{\alpha},\tilde{\beta}})}(\gamma) = f(g) + r_{\tilde{\alpha}}(\gamma)\Delta_2 + r_{\tilde{\beta}}(\gamma)\Delta_1 + r_{\tilde{\alpha}}(\gamma)r_{\tilde{\beta}}(\gamma)(\Delta - \Delta_1 - \Delta_2)$, $r_{\tilde{\alpha}}(\gamma) = \alpha + \delta(1-\gamma)$, $r_{\tilde{\beta}}(\gamma) = \beta + \omega(1-\gamma)$, $\gamma \in (0,1]$.

The example of contour map of function $r_{f(g_{\tilde{\alpha},\tilde{\beta}})}(\gamma)$ for $\gamma = 0.9$, $\delta = \omega = 0.3$, $f(g) = 0.2$, $f(g_1) = f(g_2) = 0.4$, is shown on Figure 4.

6 Conclusion

The qualitative characteristics of the combining evidence with the help of Dempster's rule with discounting were studied in this paper in the framework of Dempster-Shafer theory. In particular we found conditions on the discount rates for which the amount of ignorance after applying Dempster's rule is not increased, i.e. this rule will be still optimistic in spite of unreliable information sources. This problem was solved in general case of conflicting evidences and crisp discounting rates as well as in the case of non-conflicting evidences and fuzzy discounting rates. In addition, the problem of finding such discount rates for which a conflict of evidence will not be greater than a certain threshold and the quality of ignorance after the combination will not increase was formulated and solved.

Acknowledgements. The financial support from the Government of the Russian Federation within the framework of the implementation of the 5-100 Programme Roadmap of the National Research University Higher School of Economics is acknowledged. This work was supported by the grant 14-07-00189 of RFBR (Russian Foundation for Basic Research).

References

[1] Adamo JM (1980) Fuzzy decision trees. Fuzzy Sets and Syst 4:207–219

[2] Bronevich A, Lepskiy A (2015) Imprecision Indices: Axiomatic, Properties and Applications. Int J of General Systems 44(7-8):812–832

[3] Bronevich A, Rozenberg I (2015) The Choice of Generalized DempsterShafer Rules for Aggregating Belief Functions. Int J of Approximate Reasoning 56-A:122–136

[4] Cassels, J.W.S.: An Introduction to the Geometry of Numbers. Springer-Verlag (1959)

[5] Dempster AP (1967) Upper and Lower Probabilities Induced by a Multivalued Mapping. The Ann of Math Stat 38(2):325–339

[6] Dubois D, Prade H (1985) A Note on Measures of Specificity for Fuzzy Sets. Int J of General Systems 10:279–283

[7] Dubois D, Prade H (1986) A Set-Theoretic View of Belief Functions Logical Operations and Approximation by Fuzzy Sets. Int J of General Systems 12(3):193–226

[8] Lepskiy A (2013) About Relation between the Measure of Conflict and Decreasing of Ignorance in Theory of Evidence. In: Proc of the 8th Conf of the Eur Soc for Fuzzy Logic and Tech. Atlantis Press, Amsterdam, 355–362

[9] Lepskiy A (2014) General Schemes of Combining Rules and the Quality Characteristics of Combining. In: BELIEF 2014, LNAI 8764, Springer, 29–38

[10] Pichon F, Denoeux T (2009) Interpretation and Computation of alpha-Junctions for Combining Belief Functions. In: 6th Int Symp on Imprec Prob: Theor and Appl (ISIPTA '09), Durham, UK

[11] Sentz K, Ferson S (2002) Combination of Evidence in Dempster-Shafer Theory. In: Report SAND 2002-0835, Sandia Nat Labor

[12] Shafer G (1976) A Mathematical Theory of Evidence. Princeton Univ Press, Princeton

[13] Smets Ph (1995) The Canonical Decomposition of Weighted Belief. In: Int Joint Conf on Artif Intel 1896–1901

[14] Wang, X., Ruan, D., Kerre, E.E.: Mathematics of Fuzziness Basic Issues. Berlin: Springer (2009)