# General Schemes of Combining Rules and the Quality Characteristics of Combining

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**Abstract.** Some general schemes and examples of aggregation of two belief functions into a single belief function are considered in this paper. We find some sufficient conditions of change of ignorance when evidences are combined with the help of various rules. It is shown that combining rules can be regarded as pessimistic or optimistic depending on the sign of the change of ignorance after applying.

**Keywords:** combining rules, change of ignorance.

## 1 Introduction

The study of combining rules of evidence is one of the central directions of research in the belief function theory. The combining rule can be considered as an operator which aggregates the information obtained from different sources. The review of some popular combining rules can be found in [14].

This paper has two purposes. The first purpose is research of general schemes of combining of evidences. We can consider the combining rule as a special type of aggregation function [9] $\varphi : Bel^2(X) \to Bel(X)$, where $Bel(X)$ be a set of all belief functions on finite set $X$. The different axioms of aggregation of information obtained from different sources are considered (see, for example, [16], [4], [11], [10]). Some general schemes and examples of aggregation of two belief functions into a single belief function are given in Section 4.

The second purpose is research of quality characteristics of combining. These characteristics can be divided into a priori characteristics that estimate the quality of information sources and a posteriori characteristics which estimate the result of combining. The following characteristics are relevant to the first group: a) the reliability of sources in discount rule [15]; b) the conflict measure of evidence [12] in Dempster's rule, Yage's rule [17] etc.; c) the degree of independence of evidence. The amount of change of ignorance after the use of combining rule is the most important a posteriori characteristic. The amount of ignorance that contained in evidence can be estimated with the help of imprecision indices [3]. The generalized Hartley measure is an example of such index [6]. Some sufficient

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conditions of change of ignorance when evidences are combined with the help of various rules are described in Section 5.

We have to take into account not only aggregated evidences but also who combines this evidence. For example, let we have information from two sources about prognosticated share price. Let the first source predicted that share price will be in an interval $A_1$ and the second source predicted share price in an interval $A_2$. If a pessimist aggregates the information from two sources then he will predict the share price in the set $A_1 \cup A_2$. But if an optimist aggregates the information then he will predict the share price in the set $A_1 \cap A_2$. In other words, decision maker applies the different combining rules in depending on the price of a wrong decision, an own caution and other factors. It is known that some combining rules (for example, Dubois and Prade's disjunctive consensus rule [8]) have a pessimistic character in the sense that amount of ignorance does not decrease after their applying. The other rules are optimistic because the amount of ignorance is decreased after their applying. The majority of rules have the mixed type because their character depends on a posteriori characteristics of information sources. In Section 6 it is shown that level of optimism or pessimism in combining rule can be estimated numerically with the help of imprecision indices.

## 2 Basic Definitions and Notation

The notion of belief function is the main notion of Dempster-Shafer theory (evidence theory). Let $X$ be a finite universal set, $2^X$ is a set of all subsets of $X$. We will consider the belief function (or belief measure) [15] $g : 2^X \to [0,1]$. The value $g(A)$, $A \in 2^X$, is interpreted as a degree of confidence that the true alternative of $X$ belongs to set $A$. A belief function $g$ is defined with the help of set function $m_g(A)$ called the basic probability assignment (bpa). This function should satisfy the following conditions [15]: $m_g : 2^X \to [0,1]$, $m_g(\emptyset) = 0$, $\sum_{A \subseteq X} m_g(A) = 1$. Then

$$g(A) = \sum_{B:\, B \subseteq A} m_g(B).$$

Let the set of all belief measures on $2^X$ be denoted by $Bel(X)$ and the set of all set functions on $2^X$ be denoted by $M(X)$.

The belief function $g \in Bel(X)$ can be represented with the help of so called categorical belief functions $\eta_{\langle B \rangle}(A) = \begin{cases} 1, B \subseteq A, \\ 0, B \not\subseteq A, \end{cases}$ $A \subseteq X$ $B \neq \emptyset$. Then $g = \sum_{B \in 2^X \setminus \{\emptyset\}} m_g(B) \eta_{\langle B \rangle}$. The subset $A \in 2^X$ is called a focal element if $m_g(A) > 0$. Let $\mathcal{A}(g)$ be the set of all focal elements related to the belief function $g$. The pair $F(g) = (\mathcal{A}(g), m_g)$ is called a body of evidence. Let $\mathcal{F}(X)$ be the set of all bodies of evidence on $X$.

## 3 Combining Rules

We consider below only a few basic combining rules.

a) Dempster's rule. This rule was introduced in [5] and generalized in [15] for combining arbitrary independent evidence. This rule is defined as

$$m_D(A) = \frac{1}{1-K} \sum_{A_1 \cap A_2 = A} m_{g_1}(A_1) m_{g_2}(A_2), \quad A \neq \emptyset, \quad m_D(\emptyset) = 0, \tag{1}$$

$$K = K(g_1, g_2) = \sum_{A_1 \cap A_2 = \emptyset} m_{g_1}(A_1) m_{g_2}(A_2). \tag{2}$$

The value $K(g_1, g_2)$ characterizes the amount of conflict in two information sources which defined with the help of bodies of evidence $F(g_1)$ and $F(g_2)$. If $K(g_1, g_2) = 1$ then it means that information sources are absolutely conflicting and Dempster's rule cannot be applied.

The discount of bpa was introduced by Shafer [15] for accounting of reliability of information:

$$m^\alpha(A) = (1-\alpha) m(A), \quad A \neq X, \quad m^\alpha(X) = \alpha + (1-\alpha) m(X). \tag{3}$$

The coefficient $\alpha \in [0, 1]$ characterizes the degree of reliability of information. If $\alpha = 0$ then it means that information source is absolutely reliable. If $\alpha = 1$ then it means that information source is absolutely non-reliable. The Dempster's rule (2) applies after discounting of bpa of two evidences. This modification often called the discount rule.

b) Yager's modified Dempster's rule. This rule was introduced in [17] and it is defined as

$$q(A) = \sum_{A_1 \cap A_2 = A} m_{g_1}(A_1) m_{g_2}(A_2), \quad A \in 2^X, \tag{4}$$

$$m_Y(A) = q(A), \quad A \neq \emptyset, X, \quad m_Y(\emptyset) = 0, \quad m_Y(X) = q(\emptyset) + q(X). \tag{5}$$

c) Zhang's center combination rule. This rule was introduced in [18] and it is defined as

$$m_Z(A) = \sum_{A_1 \cap A_2 = A} r(A_1, A_2) m_{g_1}(A_1) m_{g_2}(A_2), \quad A \in 2^X,$$

where $r(A_1, A_2)$ is a measure of intersection of sets $A_1$ and $A_2$.

d) Dubois and Prade's disjunctive consensus rule [8]:

$$m_{DP}(A) = \sum_{A_1 \cup A_2 = A} m_{g_1}(A_1) m_{g_2}(A_2), \quad A \in 2^X. \tag{6}$$

Any combining rule of two bodies of evidence induces aggregation of two belief functions which correspond to these bodies.

## 4 Combining Rule both the Aggregation of Evidence

We will consider an operator $\varphi : Bel^2(X) \to Bel(X)$ that is called the aggregation of two belief functions $g_1, g_2 \in Bel(X)$ in one belief function $g = \varphi(g_1, g_2) \in Bel(X)$. The vector of bpa $(m_g(B))_{B\subseteq X}$ corresponds bijective (with the help of Möbius transform) to belief function $g \in Bel(X)$ if we define some ordering of all subsets of the universal set $X$: $g \leftrightarrow \mathbf{m}_g = (m_g(B))_{B\subseteq X}$. Therefore there is an aggregation of bpa $\mathbf{m}_g = \Phi(\mathbf{m}_{g_1}, \mathbf{m}_{g_2})$ for any aggregation of belief functions $g = \varphi(g_1, g_2)$ and vice versa. We consider some special cases of aggregation of belief functions.

**1. Pointwise Aggregation of Belief Functions.** The new value of belief function $g(A) = \varphi(g_1(A), g_2(A))$ is associated with every pair $(g_1(A), g_2(A))$ of belief functions on the same set $A \in 2^X$. In this case the aggregation operator $\varphi$ is a function $\varphi : [0,1]^2 \to [0,1]$ which must satisfy the special conditions for preserving total monotonicity of resulting set function. These conditions can be formulated in terms of finite differences, defined with the help of the following constructions: if $\Delta\mathbf{x}_1, ..., \Delta\mathbf{x}_s \in [0,1]^2$ $(\mathbf{x}+\Delta\mathbf{x}_1+...+\Delta\mathbf{x}_k \in [0,1]^2$ for all $k = 1, ..., s)$ then $\Delta^s\varphi(\mathbf{x}; \Delta\mathbf{x}_1, ..., \Delta\mathbf{x}_s) = \sum\limits_{k=0}^{s} (-1)^{s-k} \sum\limits_{1\le i_1<...<i_k\le s} \varphi\left(\mathbf{x} + \Delta\mathbf{x}_{i_1} + ... + \Delta\mathbf{x}_{i_k}\right)$ (if $k = 0$ then appropriate summand is equal $(-1)^s\varphi(\mathbf{x})$).

**Theorem 1.** *[1], [2]. The function $\varphi :\ [0,1]^2 \to [0,1]$ defines the aggregation operator of belief functions by the rule $g(A) = \varphi(g_1(A), g_2(A))$, $A \in 2^X$, $g_1, g_2 \in Bel(X)$ iff it satisfies the conditions:*

1. $\varphi(\mathbf{0}) = 0$, $\varphi(\mathbf{1}) = 1$;
2. $\Delta^k\varphi(\mathbf{x}; \Delta\mathbf{x}_1, ..., \Delta\mathbf{x}_k) \ge 0$, $k = 1, 2, ...$ *for all* $\mathbf{x}, \Delta\mathbf{x}_1, ..., \Delta\mathbf{x}_k \in [0;1]^2$, $\mathbf{x} + \Delta\mathbf{x}_1 + ... + \Delta\mathbf{x}_k \in [0,1]^2$.

**2. Pointwise Aggregation of BPA.** The new bpa $m_g(A) = \Phi(m_{g_1}(A), m_{g_2}(A))$ is associated with every pair $(m_{g_1}(A), m_{g_2}(A))$ of bpa for all $A \in 2^X$. Note that this aggregation was considered in [13] in the case of probability measures and it was called Strong Stepwise Function Property.

**Theorem 2.** *The continuous function $\Phi :\ [0,1]^2 \to [0,1]$ defines the aggregation operator of bpa by the rule $m_g(A) = \Phi(m_{g_1}(A), m_{g_2}(A))$, $A \in 2^X$, $g_1, g_2 \in Bel(X)$ iff it satisfies the condition $\Phi(s,t) = \lambda s + (1-\lambda)t$, $\lambda \in [0,1]$.*

*Proof.* We prove this result for $X = \{x_1, x_2\}$ without loss of generality. Let $S = \{\mathbf{x} = (x_i) :\ \sum_i x_i = 1,\ x_i \in [0,1]\ \forall i\}$. Then function $\Phi :\ [0,1]^2 \to [0,1]$ defines the above operator of aggregation satisfying the condition: if $\mathbf{x} = (x_i)$, $\mathbf{y} = (y_i) \in S$ and $\Phi(x_i, y_i) = z_i$, then $\mathbf{z} = (z_i) \in S$. We have for $\mathbf{x} = (\alpha, r-\alpha, 1-r)$, $\mathbf{y} = (p, p, 1-2p) \in S$

$$\Phi(\alpha, p) + \Phi(r-\alpha, p) + \Phi(1-r, 1-2p) = 1, \tag{7}$$

where $\alpha, r, r-\alpha \in [0,1]$, $p \in [0, \frac{1}{2}]$. On the other side the following equality

$$\Phi(r, p) + \Phi(1-r, 1-p) = 1 \tag{8}$$

is true for $\mathbf{x} = (r, p, 0)$, $\mathbf{y} = (1-r, 1-p, 0) \in S$. Then we have from (7) and (8)

$$\Phi(\alpha, p) + \Phi(r-\alpha, p) = \Phi(r, p) + \Phi(1-r, 1-p) - \Phi(1-r, 1-2p).$$

If we take $p = 0$ in last equality then the following equation is true $\Phi(\alpha, 0) + \Phi(r-\alpha, 0) = \Phi(r, 0)$. If $r - \alpha = \beta$ then last equality can be rewritten as $\Phi(\alpha, 0) + \Phi(\beta, 0) = \Phi(\alpha+\beta, 0)$. In other words, the function $\Phi(s, 0)$ satisfies Cauchy's functional equation on $[0, 1]$. It is known that if a continious function satisfies Cauchy's functional equation then it is an additive function: $\Phi(s, 0) = k_1 s$, $s \in [0, 1]$ and $k_1 \in [0, 1]$ because $\Phi(s, 0) \in [0, 1]$ for all $s \in [0, 1]$. By analogy $\Phi(0, t) = k_2 t$, $t \in [0, 1]$, $k_2 \in [0, 1]$. Now we get from (7) for $r = 1$, $p = 0$

$$\Phi(1, 0) + \Phi(0, 1) = k_1 + k_2 = 1. \tag{9}$$

If $\mathbf{x} = (1-\alpha, 0, \alpha)$, $\mathbf{y} = (0, 1-\beta, \beta) \in S$, $\alpha, \beta \in [0, 1]$ then we have $\Phi(1-\alpha, 0) + \Phi(0, 1-\beta) + \Phi(\alpha, \beta) = 1$. Thus $\Phi(\alpha, \beta) = 1 - \Phi(1-\alpha, 0) - \Phi(0, 1-\beta) = 1 - k_1(1-\alpha) - k_2(1-\beta) = k_1\alpha + k_2\beta$, with account of (9).

This result is a generalization of the corresponding result for probability measures [13].

**3. Bilinear Aggregation of Belief Functions.** In this case the aggregation function $\varphi$ should be linear for each argument so

$$\varphi(\alpha g_1 + (1-\alpha) g_2, g_3) = \alpha\varphi(g_1, g_3) + (1-\alpha)\varphi(g_2, g_3), \quad \alpha \in [0, 1]. \tag{10}$$

Since we have $g_i = \sum_{B \in 2^X \setminus \{\emptyset\}} m_{g_i}(B)\eta_{\langle B \rangle} \in Bel(X)$, $i = 1, 2$, then every bilinear function on $Bel^2(X)$ has the form

$$\varphi(g_1, g_2) = \sum_{A, B \in 2^X \setminus \{\emptyset\}} m_{g_1}(A) m_{g_2}(B) \gamma_{A,B}, \tag{11}$$

where $\gamma_{A,B} = \varphi\left(\eta_{\langle A \rangle}, \eta_{\langle B \rangle}\right)$ is some set function on $2^X$.

We consider the non-empty set $\mathcal{B}(X) \subseteq Bel^2(X)$ which satisfies the condition: if $(g_1, g_2) \in \mathcal{B}(X)$ then $\left(\eta_{\langle A \rangle}, \eta_{\langle B \rangle}\right) \in \mathcal{B}(X)$ for all $A \in \mathcal{A}(g_1)$, $B \in \mathcal{A}(g_2)$.

**Theorem 3.** *The bilinear set function $\varphi : \mathcal{B}(X) \to M(X)$ of the form (11) determines the belief function iff $\gamma_{A,B} = \varphi\left(\eta_{\langle A \rangle}, \eta_{\langle B \rangle}\right) \in Bel(X)$ for all $\left(\eta_{\langle A \rangle}, \eta_{\langle B \rangle}\right) \in \mathcal{B}(X)$.*

The Dubois and Prade's disjunctive consensus rule and Dempster's rule (Yager's rule) for non conflicting evidences are the examples of bilinear aggregation functions of the form (11).

**4. Bilinear Normalized Aggregation of Belief Functions.** We consider the aggregation function of belief measures of the form

$$\varphi_0(g_1, g_2) = \frac{\varphi(g_1, g_2)}{\varphi(g_1, g_2)(X)}, \tag{12}$$

where $\varphi(g_1, g_2)$ is a bilinear aggregation function which satisfies the condition (10). We will consider that $\gamma_{A,B}(C) \geq 0$ for all $A, B, C \in 2^X \backslash \{\emptyset\}$. It is obvious that aggregation function $\varphi_0$ cannot be determined on the whole set $Bel^2(X)$. The function $\varphi_0$ will be determined on the set

$$\mathcal{B}_\varphi(X) = \left\{(g_1, g_2) \in Bel^2(X) | \ \exists \ A_i \in \mathcal{A}(g_i), \ i = 1, 2 : \ \varphi\left(\eta_{\langle A_1 \rangle}, \eta_{\langle A_2 \rangle}\right)(X) \neq 0\right\}$$

that follows from (11).

**Theorem 4.** *Let $\varphi$ be a bilinear aggregation function which satisfies the condition (10). The function $\varphi_0 : \mathcal{B}_\varphi(X) \to M(X)$ of the form (12) determines the belief function iff $\gamma_{A,B}/\gamma_{A,B}(X) \in Bel(X)$, $\gamma_{A,B} = \varphi\left(\eta_{\langle A \rangle}, \eta_{\langle B \rangle}\right)$ for all $\left(\eta_{\langle A \rangle}, \eta_{\langle B \rangle}\right) \in \mathcal{B}_\varphi(X)$.*

The Dempster's rule and Zhang's center combination rule are the examples of bilinear normalized aggregation functions of the form (12):

## 5 Change of Ignorance when Evidences are Combined

Let we have two sources of information, and this information is described by belief functions $g_1, g_2 \in Bel(X)$ respectively. Let some rule $\varphi$ be used for combining of these belief functions. We will get the new belief function $g = \varphi(g_1, g_2) \in Bel(X)$. The different information characteristics of aggregation of belief functions were studied in a number of works (see [7]). Below we consider only one aspect associated with change of information uncertainty. The measure of information uncertainty associated to the each belief function. Then we have a question about change of this measure after combining of evidence. There are some approaches for defining uncertainty measures in evidence theory. We will follow the approach which was considered in [3]. This approach is based on the notion of imprecision index.

Let we know only that true alternative belongs to the non-empty set $B \subseteq X$. This situation can be described with the help of primitive belief measure $\eta_{\langle B \rangle}(A)$, $A \subseteq X$, which gives the lower probability of an event $x \in A$. The degree of uncertainty of such measure is described by the well-known Hartley's measure $H(\eta_{\langle B \rangle}) = \log_2 |B|$. There is the generalization of Hartley's measure. Let $g$ be a belief function that can be represented by $g = \sum_{B \in 2^X \backslash \{\emptyset\}} m_g(B)\eta_{\langle B \rangle} \in Bel(X)$. Then the generalized Hartley's measure is defined by [6] $GH(g) = \sum_{B \in 2^X \backslash \{\emptyset\}} m_g(B) \log_2 |B|$.

**Definition 1.** *[3]. A functional $f : Bel(X) \to [0, 1]$ is called an imprecision index if the following conditions are fulfilled:*

1. *if $g$ is a probability measure then $f(g) = 0$;*
2. *$f(g_1) \geq f(g_2)$ for all $g_1, g_2 \in Bel(X)$ where $g_1 \leq g_2$ (i.e. $g_1(A) \leq g_2(A)$ for all $A \in 2^X$);*
3. *$f\left(\eta_{\langle X \rangle}\right) = 1$.*

We call the imprecision index strict if $f(g) = 0 \Leftrightarrow g$ is a probability measure. The imprecision index $f$ on $Bel(X)$ is called linear (lii) if for any linear combination $\sum_{j=1}^{k} \alpha_j g_j \in Bel(X)$, $\alpha_j \in \mathbb{R}$, $g_j \in Bel(X)$, $j = 1, ..., k$, we have $f\left(\sum_{j=1}^{k} \alpha_j g_j\right) = \sum_{j=1}^{k} \alpha_j f(g_j)$.

Since any linear functional $f$ on $Bel(X)$ is defined uniquely by its values on a set of primitive measures $\{\eta_{\langle B\rangle}\}_{B \in 2^X \setminus \{\emptyset\}}$, then it allows us to define $f$ with the help of set function $\mu_f : 2^X \to \mathbb{R}$ by the rule $\mu_f(B) = f\left(\eta_{\langle B\rangle}\right)$, $B \in 2^X \setminus \{\emptyset\}$. We set by definition that $\mu_f(\emptyset) = 0$ for every imprecision index $f$.

**Proposition 1.** *[3]. A functional $f : Bel(X) \to [0,1]$ is a lii on $Bel(X)$ iff $f(g) = \sum_{B \in 2^X \setminus \{\emptyset\}} m_g(B)\mu_f(B)$, where set function $\mu_f$ satisfies the conditions:*

1. *$\mu_f(\{x\}) = 0$ for all $x \in X$;*
2. *$\mu_f(X) = f\left(\eta_{\langle X\rangle}\right) = 1$;*
3. *$\sum_{B: A \subseteq B} (-1)^{|B \setminus A|} \mu_f(B) \leq 0$ for all $A \neq \emptyset, X$.*

Now we are going to give some sufficient conditions for the different rules under which the amount of ignorance decreases or increases after combining. The first result is well known [7].

**Proposition 2.** *If $g = \varphi_{DP}(g_1, g_2)$, $g_1, g_2 \in Bel(X)$, where $\varphi_{DP}$ is the Dubois and Prade's disjunctive consensus rule (6), then inequalities $f(g) \geq f(g_i)$, $i = 1, 2$ are true for any lii $f$.*

**Proposition 3.** *Let $g_1, g_2$ be such belief measures that their conflict measure $K(g_1, g_2) = 0$ and $g = \varphi_{\alpha,\beta}(g_1, g_2)$, where $\varphi_{\alpha,\beta}$ is a Dempster's rule (1) after applied of discount rule (3) to the $g_1, g_2$ with coefficients $\alpha$, $\beta \in [0,1]$ correspondingly. If the inequality $\alpha\beta + (1-\alpha)\beta m_{g_1}(X) + \alpha(1-\beta)m_{g_2}(X) \leq (\alpha + \beta - \alpha\beta)f(g_i)$, is true for lii $f$ then $f(g) \leq f(g_i)$, $i = 1, 2$.*

The last Proposition shows that the amount of ignorance is decreased obviously after combining of evidence with the help of discount rule if ignorance of initial evidence were largish.

**Proposition 4.** *Let $g_1, g_2$ be such belief measures that their conflict measure (see formula (2)) $K = K(g_1, g_2)$ satisfies the condition $K + m_{g_1}(X)m_{g_2}(X) \leq m_{g_i}(X)$, $i = 1, 2$, $g = \varphi_Y(g_1, g_2)$, where $\varphi_Y$ is a Yager's rule (4)-(5). Then the inequalities $f(g) \leq f(g_i)$, $i = 1, 2$ are true for any lii $f$.*

The value $m_{g_1}(X)$ characterizes the imprecision of information given by function $g_1$. Therefore the condition $K + m_{g_1}(X)m_{g_2}(X) \leq m_{g_1}(X) \Leftrightarrow K \leq m_{g_1}(X)(1 - m_{g_2}(X))$ in Proposition 4 means that the amount of ignorance can be decreased with the help of Yager's rule if the conflict between the evidences is not very large with respect to amount of ignorance.

**Corollary 1.** *Let $g_1, g_2$ be such belief measures that their conflict measure (see formula (2)) $K(g_1, g_2) = 0$, $g = \varphi(g_1, g_2)$, where $\varphi$ is Dempster's rule (1). Then the inequalities $f(g) \leq f(g_i)$, $i = 1, 2$ are true for any lii $f$.*

This corollary shows that the imprecision of information is not increased if we aggregate information from many non-conflict sources with the help of Dempster's rule (Yager's rule). If we have conflicting information sources ($K > 0$) then resulting evidence can have a larger imprecision than the imprecision of sources (see [12]). But we can formulate the following sufficient condition of decreasing of ignorance for Dempster's rule and conflicting ($K > 0$) information sources.

Let $C$ be the smallest number satisfying the inequality $\mu_f(A_1 \cap A_2) \le C\mu_f(A_1)\mu_f(A_2)$ for all $A_i \in \mathcal{A}(g_i)$, $i = 1, 2$. Note that $\min\limits_{A:\mu_f(A)>0} \mu_f(A) \le \frac{1}{C}$. Moreover $C \ge 1$ if belief functions $g_1, g_2$ are not probability measures and $f$ is a strict lii.

**Proposition 5.** *Let $g_1, g_2$ are such belief measures that their conflict measure $K = K(g_1, g_2) \ne 1$ satisfies the condition $K \le 1 - Cf(g_2)$ ($K \le 1 - Cf(g_1)$), $g = \varphi_D(g_1, g_2)$, where $\varphi_D$ is a Dempster's rule (1). Then inequality $f(g) \le f(g_1)$ ($f(g) \le f(g_2)$) is true for any strict lii $f$.*

## 6 Pessimistic and Optimistic Combining Rules

Let we have two sources of information, and this information is described by primitive belief functions $\eta_{\langle A\rangle}$ and $\eta_{\langle B\rangle}$ respectively, where $A, B \in 2^X \backslash \{\emptyset\}$. The first source states that true alternative is contained in set $A$, but second source states that true alternative is contained in set $B$.

If we apply the Dubois and Prade's disjunctive consensus rule for these primitive belief functions then we will get $\varphi_{DP}(\eta_{\langle A\rangle}, \eta_{\langle B\rangle}) = \eta_{\langle A\cup B\rangle}$. By other words we got the statement that a true alternative is contained in set $A \cup B$. This statement can be considered as more pessimistic than an initial statement because uncertainty does not decreased after combining. For example, if lii of initial measures was equal to $f(\eta_{\langle A\rangle}) = \mu_f(A)$ and $f(\eta_{\langle B\rangle}) = \mu_f(B)$ respectively, then this index is equal to $f(\eta_{\langle A\cup B\rangle}) = \mu_f(A \cup B) \ge f(\eta_{\langle A\rangle})$ for resulting measure.

If we apply the Dempster's rule for these primitive belief functions then we will get $\varphi_D(\eta_{\langle A\rangle}, \eta_{\langle B\rangle}) = \eta_{\langle A\cap B\rangle}$ for $A \cap B \ne \emptyset$. We got the statement after combining that a true alternative is contained in set $A\cap B$. This statement can be considered to be more optimistic than the initial statement because uncertainty does not increased after combining: $f(\eta_{\langle A\cap B\rangle}) = \mu_f(A \cap B) \le f(\eta_{\langle A\rangle})$.

If we apply the discount rule for these two primitive belief functions with parameters $\alpha$, $\beta \in [0, 1]$ respectively, then we will get new measures after discounting $\eta^{(\alpha)}_{\langle A\rangle} = (1-\alpha)\eta_{\langle A\rangle} + \alpha\eta_{\langle X\rangle}$, $\eta^{(\beta)}_{\langle B\rangle} = (1-\beta)\eta_{\langle B\rangle} + \beta\eta_{\langle X\rangle}$. Let $A \cap B \ne \emptyset$. Then the conflict $K = 0$ and we get resultant measure after application of Dempster's rule to new discounting measures:

$$g_{\alpha,\beta} = \varphi_D\left(\eta^{(\alpha)}_{\langle A\rangle}, \eta^{(\beta)}_{\langle B\rangle}\right) =$$

$$(1-\alpha)(1-\beta)\eta_{\langle A\cap B\rangle} + (1-\alpha)\beta\eta_{\langle A\rangle} + \alpha(1-\beta)\eta_{\langle B\rangle} + \alpha\beta\eta_{\langle X\rangle}. \tag{13}$$

We will suppose that the information sources are sufficiently reliable. Then $\alpha, \beta \approx 0$. In this case we will get the following resulting measure instead of (13) if we neglect members of second order of $\alpha$ and $\beta$

$$g_{\alpha,\beta} = \varphi_D\left(\eta_{\langle A\rangle}^{(\alpha)}, \eta_{\langle B\rangle}^{(\beta)}\right) = (1-\alpha-\beta)\eta_{\langle A\cap B\rangle} + \beta\eta_{\langle A\rangle} + \alpha\eta_{\langle B\rangle}.$$

The linear imprecision index of this measure is equal to $f(g_{\alpha,\beta}) = (1-\alpha-\beta)\mu_f(A\cap B) + \beta\mu_f(A) + \alpha\mu_f(B)$. It is easy to see that in this case we can get different relations between the indices $f(g_{\alpha,\beta})$ and $f(\eta_{\langle A\rangle}) = \mu_f(A)$, $f(\eta_{\langle B\rangle}) = \mu_f(B)$ depending on the choice $\alpha$ and $\beta$. In particular, we have

$$\begin{cases} f(g_{\alpha,\beta}) \le f(\eta_{\langle A\rangle}), \\ f(g_{\alpha,\beta}) \le f(\eta_{\langle B\rangle}) \end{cases} \Leftrightarrow \alpha\Delta(B,A) + \beta\Delta(A,B) \le \min\left\{\Delta(A,B), \Delta(B,A)\right\},$$

where $\Delta(A,B) = \mu_f(A) - \mu_f(A\cap B)$.

From last estimations we can make the following conclusion. If the degree of reliability of information sources is large (i.e. $\alpha \approx 0$, $\beta \approx 0$) then discount rule will act as optimistic rule. Otherwise, when the information sources are non reliable ($\alpha$ and $\beta$ are large) then discount rule will be act as pessimistic rule.

## 7 Conclusion

In this paper we consider some general schemes and examples of aggregation of two belief functions into one belief function. The well-known combining rules are obtained from these general schemes in particular cases. Furthermore, an important a posteriori characteristic of quality of combining like a change of ignorance after the use of combining rule is considered. This value is estimated in this paper with the help of linear imprecision indices.

Some sufficient conditions of change of ignorance after applying of different combining rules are found. In particular we show that amount of ignorance do not decrease after using of Dubois and Prade's disjunctive consensus rule. In contrast the amount of ignorance does not increase after using of Dempster's rule for two non-conflict evidences.

In this sense these rules can be considered as a pessimistic rule and optimistic rule correspondingly. At the same time, the discount rule can be the pessimistic rule or the optimistic rule depending of values of reliability coefficients of information sources. The sufficient conditions on reliability coefficients of this rule to be pessimistic or optimistic were found.

## References

1. Bronevich, A.G.: On the closure of families of fuzzy measures under eventwise aggregations. Fuzzy sets and systems **153**, 45–70 (2005)
2. Bronevich, A., Lepskiy, A.: Operators for convolution of fuzzy measures. In: Grzegorzewski, P. et al. (eds.) Soft Methods in Probability, Statistics and Data Analysis, pp. 84-91. Physical-Verl., Heidelberg (2002)

3. Bronevich, A., Lepskiy, A.: Measuring uncertainty with imprecision indices. In: Proc. of the Fifth International Symposium on Imprecise Probabilities and Their Applications (ISIPTA"07), pp. 47-56. Prague, Czech Republic (2007)
4. Cheng, Y., Kashyap, R.L.: An axiomatic approach for combining evidence from a variety of sources. Journal of Intelligent and Robotic Systems **1(1)**, 17–33 (1988)
5. Dempster, A.P.: Upper and Lower Probabilities Induced by a Multivalued Mapping. The Annals of Mathematical Statistics **38(2)**, 325–339 (1967)
6. Dubois, D., Prade, H.: A note on measures of specificity for fuzzy sets. Int. J. of General Systems **10**, 279–283 (1985)
7. Dubois, D., Prade, H.: A set-theoretic view of belief functions Logical operations and approximation by fuzzy sets. Int. J. of General Systems **12(3)**, 193–226 (1986)
8. Dubois, D., Prade, H.: On the combination of evidence in various mathematical frameworks. In: Flamm, J., Luisi, T. (eds.) Reliability Data Collection and Analysis, pp. 213-241. Brussels, ECSC, EEC, EAFC (1992)
9. Grabisch, M., Marichal, J.L., Mesiar, R., Pap, E.: Aggregation Functions. Cambridge University Press (2009)
10. Jiao, L., Pan, Q., Liang, Y., Feng, X., Yang, F.: Combining sources of evidence with reliability and importance for decision making. Central European Journal of Operations Research, December, 1–20 (2013)
11. Johnson, N.L., Kotz, S.: Axiomatic approaches to formulas for combining likelihoods or evidence. Journal of Statistical Computation and Simulation **31(1)**, 49–54 (1989)
12. Lepskiy, A.: About Relation between the Measure of Conflict and Decreasing of Ignorance in Theory of Evidence. In: Proc. of the 8th conf. of the European Society for Fuzzy Logic and Technology, pp. 355-362. Atlantis Press, Amsterdam (2013)
13. McConway, K.J.: Marginalization and Linear Opinion Pools. Journal of the American Statistical Association **76:374**, 410–414 (1981)
14. Sentz, K., Ferson, S.: Combination of evidence in Dempster-Shafer theory. In: Report SAND 2002-0835, Sandia National Laboratories (2002)
15. Shafer, G.: A Mathematical Theory of Evidence. Princeton University Press (1976)
16. Smets, Ph.: The Combination of Evidence in the Transferable Belief Model. IEEE Trans. PAMI **12**, 447–458 (1990)
17. Yager, R.: On the Dempster-Shafer Framework and New Combination Rules. Information Sciences **41**, 93–137 (1987)
18. Zhang, L.: Representation, independence, and combination of evidence in the Dempster-Shafer theory. In: Yager, R.R., Kacprzyk, J., Fedrizzi, M.(eds.) Advances in the Dempster-Shafer Theory of Evidence, pp. 51-69. John Wiley & Sons, New York (1994)